1500 CALCULUS PROBLEMS SOLVED

A STUDENT-CENTERED WORKBOOK

FULL SOLUTIONS AND COMPLETE NOTES

VALERIE SUN

Contents

Preface v
How To Use This Book vii

1 LIMITS AND CONTINUITY 1
Chapter 1 Notes 3
Chapter 1 Exercises 14
Defining a Limit 14
Evaluating Limits Analytically 14
Limits by Change of Variable 16
Squeeze Theorem 16
Continuity 17
Intermediate Value Theorem 18
Limits with Infinity; Vertical and Horizontal Asymptotes 18
Formal Proofs of Limits 19
Chapter 1 Exercise Solutions 21

2 DIFFERENTIATION RULES 39
Chapter 2 Notes 41
Chapter 2 Exercises 50
Basic Derivatives 50
Differentiation Formulas 51
Tangent and Normal Lines 52
Implicit Differentiation 53
Differentiating Inverse Functions 53
Logarithmic Differentiation 53
Linearization and Differentials 54
Related Rates 54
Chapter 2 Exercise Solutions 56

3 APPLICATIONS OF DIFFERENTIATION 81
Chapter 3 Notes 83
Chapter 3 Exercises 90
Mean Value Theorem and Rolle's Theorem 90
Minimum and Maximum Values 90

Using the First and Second Derivatives 91
Particle Motion 93
Indeterminate Forms and L'Hôpital's Rule 94
Newton's Method 95
Differentiation in Economics 95
Optimization 96
Chapter 3 Exercise Solutions 98

4 INTEGRATION 127
Chapter 4 Notes 129
Chapter 4 Exercises 141
Antidifferentiation 141
Definite Integrals 141
Fundamental Theorem of Calculus 144
Integration by Substitution 146
Numerical Integration 146
Chapter 4 Exercise Solutions 148

5 APPLICATIONS OF INTEGRATION 167
Chapter 5 Notes 169
Chapter 5 Exercises 176
Areas between Curves 176
Volumes with Cross Sections 177
Solids of Revolution 178
Work 179
Average Value of a Function 181
Chapter 5 Exercise Solutions 183

6 INTEGRATION TECHNIQUES 213
Chapter 6 Notes 215
Chapter 6 Exercises 225
Integration by Parts 225
Trigonometric Integrals 226
Trigonometric Substitution 227
Integration by Partial Fractions 228
Improper Integrals 228
Chapter 6 Exercise Solutions 230

7 FURTHER APPLICATIONS OF INTEGRATION 263
Chapter 7 Notes 265

Chapter 7 Exercises 272
Arc Length 272
Surface Areas of Revolution 273
Consumer Surplus and Producer Surplus 273
Moments and Centers of Mass 274
Hydrostatics 276
Probability 279
Chapter 7 Exercise Solutions 282

8 DIFFERENTIAL EQUATIONS 313
Chapter 8 Notes 315
Chapter 8 Exercises 324
Introduction to Differential Equations 324
Slope Fields and Euler's Method 324
Separation of Variables 326
Orthogonal Trajectories 327
Mixing Problems 327
Exponential Models 327
Logistic Models 329
First-Order Linear Differential Equations 331
Chapter 8 Exercise Solutions 332

9 PARAMETRIC EQUATIONS AND POLAR COORDINATES 363
Chapter 9 Notes 365
Chapter 9 Exercises 374
Parametric Equations 374
Differentiating and Integrating Parametric Functions 375
Polar Coordinates and Functions 376
Differentiating Polar Functions 377
Areas with Polar Curves 378
Additional Calculus with Parametric and Polar 378
Chapter 9 Exercise Solutions 380

10 INFINITE SEQUENCES AND SERIES 409
Chapter 10 Notes 411
Chapter 10 Exercises 426
Infinite Sequences 426
Infinite Series and Divergence Test 426
Integral Test and p-series 427
Comparison Testing 427
Alternating Series 428
Absolute Convergence and the Ratio and Root Tests 429

Power Series 429
Taylor and Maclaurin Series 430
Chapter 10 Exercise Solutions 433

Index 459

Preface

When I first studied calculus—teaching myself, without a teacher—I remember failing to absorb the textbook. Like many students, I reread sentence after sentence, convincing myself that I understood the concepts, only to be immediately stumped by the exercises section. Concepts only made sense to me after I struggled through practice problems and then *consulted the solutions*.

Doing so allowed me to identify knowledge gaps and develop lines of reasoning to cultivate conceptual understanding. Today, I use this approach to transform my students' performance and approaches to studying mathematics: They now learn by doing. Mathematics is a language, the world's most special one—the language of the universe—and no language can be acquired without learning what is right and what is wrong, and *why*. Countless scientific and technological revolutions have arisen from calculus, making mastery of the subject essential for future advancements.

Accordingly, I present *1500 Calculus Problems Solved* based on my experiences to fulfill these needs. Modern textbooks often omit solutions to exercises. They are either locked behind a paywall or, most ironically, accessible only to instructors. This choice does no favors to any aspiring mathematician, engineer, or scientist. Hence, this book closes the gap by offering practice centered around the heart of mathematics learning: sustained practice *with feedback*. Each of the 10 chapters features three sections: Notes, Exercises, and Exercise Solutions.

💡 TIP 💡
If $\sum a_n$ is alternating, then you can *disregard* the alternator when setting up L for the Ratio Test and Root Test, as both tests feature absolute value expressions.

⚠ WARNING ⚠
Do not make the following mistakes:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g'(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}.$$

Examples of tips and warnings.

Notes Sections

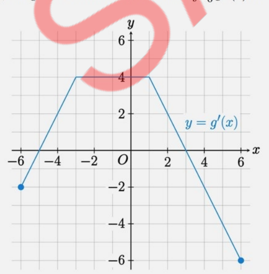
The Notes sections contain outlines of concepts, along with a summary of chapter topics and a list of studying tips. Topics are divided into subsections with key facts and equations. The following margin notes appear:

- TIPS describe any best practices and problem-solving tips.
- WARNINGS highlight common student mistakes.

Yet the Notes sections are not intended to substitute for a first-hand resource, such as an instructor's lectures. Instead, the Notes sections primarily serve to review topics, so you may find that they resemble summaries that emphasize problem-solving strategies.

85. The following figure shows the graph of $y = g'(x)$ for $-6 \leqslant x \leqslant 6$. It is known that $g(0) = 5$. Let $f(x) = \int_0^x g'(t)\,\mathrm{d}t$.

(a) Calculate $f(4)$.
(b) Find the value of $g(4)$.
(c) Evaluate $f'(4)$.
(d) Identify the x-coordinates of any critical points of f.
(e) If g'' is continuous, then find $\int_{-6}^{4} g''(x)\,\mathrm{d}x$.



An example exercise.

Exercises

In each Exercises section, there are 150 exercises that apply conceptual, algebraic, and geometric skills across all the chapter topics. Full solutions to the exercises are located at the end of each chapter. Generally, the exercises for each corresponding chapter topic are ordered in terms of difficulty. Several exercises contain multiple parts to imitate exam-style problems and synthesize chapter topics.

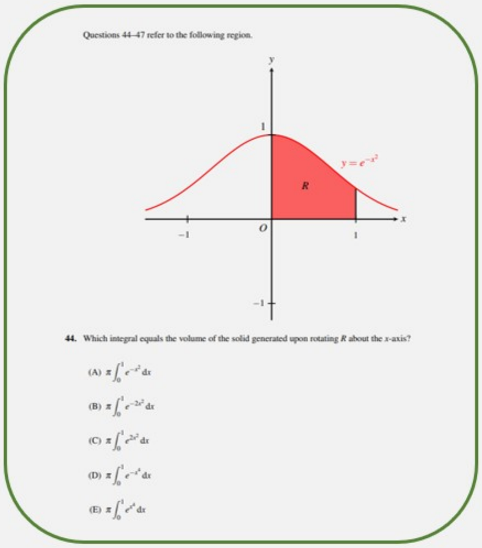
I dedicated a great deal of effort into crafting and vetting rigorous exercises that unify differing topics. Doing so forms connections, enhancing your long-term memory. For example, a problem in Chapter 7 (Further Applications of Integration) or Chapter 8 (Differential Equations)—my favorite chapters due to their rich applications of calculus—may apply ideas from Chapters 2 and 3.

The solutions in this book emphasize a problem-solving approach, teaching mathematics as a process of reasoning, not just memorizing formulas. Each solution emphasizes *how* and *why* a formula or theorem can be used, highlighting the conditions and common problem-solving fallacies.

Pedagogical Style

In the United States, most universities teach single-variable calculus in two semesters with an *early transcendentals* approach, featuring an early introduction of trigonometric, exponential, and logarithmic functions. This system better enables students to connect calculus to their chosen fields. Accordingly, this book follows the early transcendentals framework by featuring trigonometric, exponential, and logarithmic functions starting from Chapter 1.

This book is ideal for introductory, single-variable calculus courses, namely, Calculus I and Calculus II. In addition, the book is well-positioned to serve students in the Advanced Placement Calculus AB and Calculus BC courses. It can also complement curricula with a calculus component, such as International Baccalaureate mathematics courses.


Questions 44–47 refer to the following region.

44. Which integral equals the volume of the solid generated upon rotating R about the x-axis?

(A) $\pi \int_0^1 e^{-x^2}\,dx$

(B) $\pi \int_0^1 e^{-2x^2}\,dx$

(C) $\pi \int_0^1 e^{2x^2}\,dx$

(D) $\pi \int_0^1 e^{-x^4}\,dx$

(E) $\pi \int_0^1 e^{x^4}\,dx$

Part of a VALSCALC *practice exam.*

Additional Resources and Author Contact

1500 Calculus Problems Solved is part of the VALSCALC project, which I created to improve the accessibility of calculus and mathematical education. Visit valscalc.com to access a free *primary resource* for learning calculus; the website contains in-depth sections with full derivations, proofs, and animations to compliment this book's exercises. Practice exams and challenge problems with solutions are also available for all the book's chapters at no cost. These resources will enable you to continue learning and practicing calculus independently.

I first envisioned spreading the beauty of calculus at 14 years old. Years later, it is my pleasure to present *1500 Calculus Problems Solved*. I am eager to continue growing VALSCALC and improving this book. I especially welcome any suggestions or alerts of errors in any part of the book, which you may address to me at val@valscalc.com. My hope is that this book will grant you the mathematical foundations and studying habits that will guide you throughout your career—through calculus and beyond.

Valerie Sun

VALERIE SUN

How To Use This Book

Calculus is unique because each problem is different and requires a cumulative set of skills. Mathematics must be learned by actively participating, not merely reading or spectating. Therefore, to learn calculus you should feel a little discomfort, just as your muscles grow when pushed to their limits. At the same time, incorrect approaches to studying will inhibit your results, so this section aims to provide a framework for effectively using this book.

Learning Steps

The following five-step process highlights the best way to utilize this book:

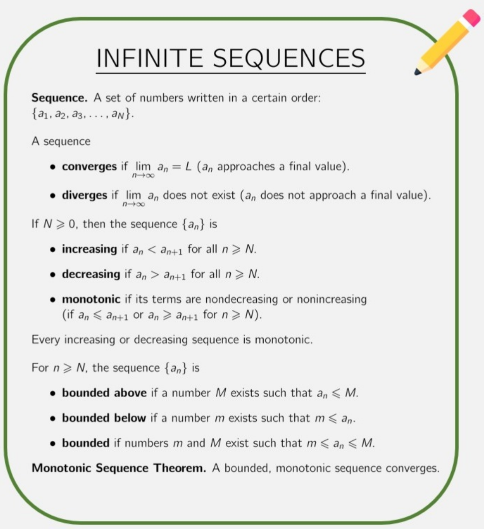
1. *Take Notes*. In the Notes section at the beginning of each chapter, write down formulas, key ideas, study tips, and warnings. Feel free to be creative in organizing the information.

2. *Attempt Exercises*. In the Exercises section, try a few problems without consulting the solutions. Spend a few minutes on each, referencing your notes as needed. At this stage, you are unlikely to complete the problems entirely; simply continue until you feel stuck.

3. *Understand the Book's Solutions*. Compare your partial solutions to the book's solutions, and note any correct procedures and partial results. At this point, focus on understanding the book's solutions: For each exercise, rework the problem (perhaps in a different color) by following the book's solution. Ask yourself: "What would I have done differently next time?" In combination with your notes, this process will help you understand the problem-solving steps.

4. *Repeat the Exercises*. Now rework each problem *without* the aid of the book's solution. To ensure deep conceptual understanding, pretend to explain or teach the elements of the solution. Doing so encourages active recall by exposing gaps in your understanding and solidifying fundamental concepts. Repeat Step 3 if you cannot yet independently produce a fully correct solution. The next time you study (or after about 30 minutes), be sure that you can fully solve the problem on your own.

5. *Try Harder Problems*. Repeat Steps 2–4 for more difficult problems in the Exercises section. Doing so builds mastery of the topic and allows you to form connections.


INFINITE SEQUENCES

Sequence. A set of numbers written in a certain order: $\{a_1, a_2, a_3, \ldots, a_N\}$.

A sequence

- **converges** if $\lim_{n\to\infty} a_n = L$ (a_n approaches a final value).
- **diverges** if $\lim_{n\to\infty} a_n$ does not exist (a_n does not approach a final value).

If $N \geqslant 0$, then the sequence $\{a_n\}$ is

- **increasing** if $a_n < a_{n+1}$ for all $n \geqslant N$.
- **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant N$.
- **monotonic** if its terms are nondecreasing or nonincreasing (if $a_n \leqslant a_{n+1}$ or $a_n \geqslant a_{n+1}$ for $n \geqslant N$).

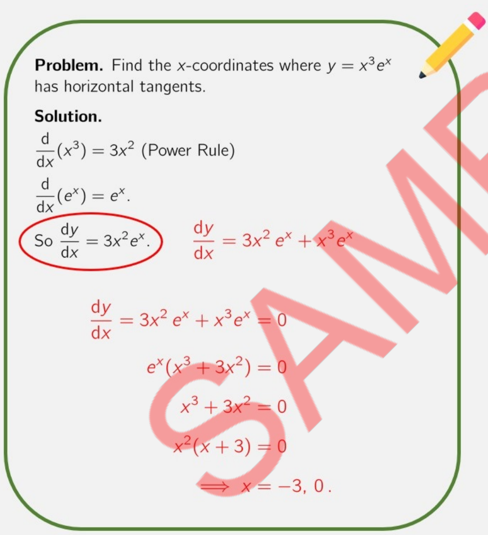
Every increasing or decreasing sequence is monotonic.

For $n \geqslant N$, the sequence $\{a_n\}$ is

- **bounded above** if a number M exists such that $a_n \leqslant M$.
- **bounded below** if a number m exists such that $m \leqslant a_n$.
- **bounded** if numbers m and M exist such that $m \leqslant a_n \leqslant M$.

Monotonic Sequence Theorem. A bounded, monotonic sequence converges.

An example set of notes (Step 1).


Problem. Find the x-coordinates where $y = x^3e^x$ has horizontal tangents.

Solution.

$\frac{d}{dx}(x^3) = 3x^2$ (Power Rule)

$\frac{d}{dx}(e^x) = e^x$.

So $\frac{dy}{dx} = 3x^2e^x$. $\quad \frac{dy}{dx} = 3x^2\,e^x + x^3e^x$

$\frac{dy}{dx} = 3x^2\,e^x + x^3e^x = 0$

$e^x(x^3 + 3x^2) = 0$

$x^3 + 3x^2 = 0$

$x^2(x+3) = 0$

$\implies x = -3,\ 0\,.$

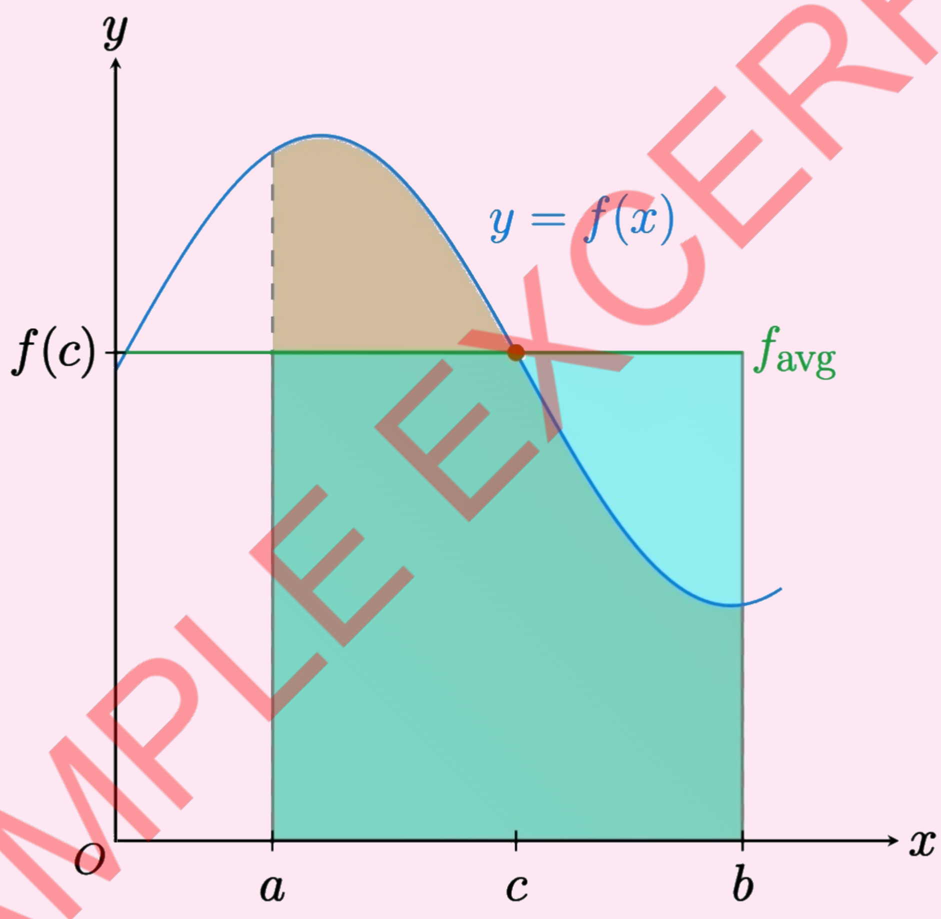
An example corrected solution (Step 3).

Additional Studying Tips

These pieces of advice help students gain the skills required to excel in a calculus class:

- Practice, practice, practice! You can never learn calculus by simply watching others solve problems.

5 APPLICATIONS OF INTEGRATION



IN THIS CHAPTER, we expand upon the utility of integrals by using them in a diverse range of applications: calculating areas between curves, volumes of solids, and work. Also, knowing a function's *average value* considerably simplifies the process of calculating an integral.

CHAPTER OBJECTIVES

By the end of this chapter, you should be able to . . .

- ☐ Set up integrals for areas bounded between curves with respect to either x or y.
- ☐ Calculate the volume of a solid whose base lies in a plane and whose cross-sections are geometric figures.
- ☐ Use the Disk Method, Washer Method, and Shell Method to calculate volumes when a bounded region is rotated about an axis.
- ☐ Calculate the work done by a force in applications involving a variable force.
- ☐ Determine the work done in stretching springs.
- ☐ Calculate the work done in expanding gases.
- ☐ Calculate the work done to pump water out of a tank.
- ☐ Define and find a function's average value on an interval.
- ☐ Define and apply the Mean Value Theorem for Integrals.

Draw a ✓ next to an objective you have mastered.

is the inner distance from the axis of revolution, then we use the Washer Method through the following formulas:

$$V = \pi \int_a^b \left([r_{\text{out}}(x)]^2 - [r_{\text{in}}(x)]^2 \right) dx, \qquad \text{[Horizontal Axis of Revolution]} \tag{5.7}$$

$$V = \pi \int_c^d \left([r_{\text{out}}(y)]^2 - [r_{\text{in}}(y)]^2 \right) dy. \qquad \text{[Vertical Axis of Revolution]} \tag{5.8}$$

As an example, the solid generated in Figure 5.5 is evaluated by using Equation (5.7). If the outer edge is f and inner edge is g, then we use $r_{\text{out}}(x) = f(x) - L$ and $r_{\text{in}}(x) = g(x) - L$. [It is also valid to use $r_{\text{out}}(x) = L - f(x)$ and $r_{\text{in}}(x) = L - g(x)$ because Equation (5.7) contains the squares of these distances.]

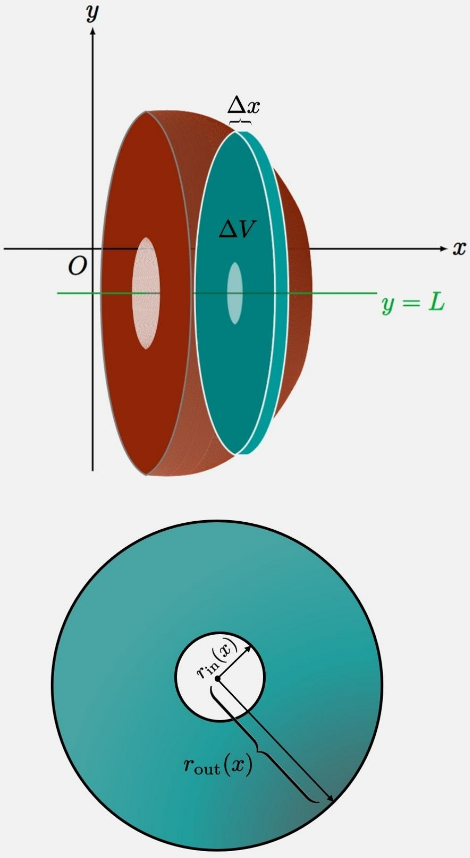


Figure 5.5

If region R is bounded between two functions $y = f(x)$ and $y = g(x)$, then the following steps enable you to calculate the volume of the solid of revolution generated by rotating R about a *horizontal line* $y = L$:

1. Sketch the bounded region R and calculate the points of intersection $x = a$ and $x = b$. Draw a vertical approximating rectangle.
2. At any x, determine the distance $r_{\text{in}}(x)$ from $y = L$ to the rectangle's closest horizontal side. Then find the distance $r_{\text{out}}(x)$ from $y = L$ to the rectangle's farthest horizontal side.
3. Use Equation (5.7) with the bounds $x = a$ and $x = b$.

If region R is bounded between two functions $x = f(y)$ and $x = g(y)$, then the following steps enable you to calculate the volume of the solid of revolution generated by rotating R about a *vertical line* $x = L$:

1. Sketch the bounded region R and calculate the points of intersection $y = c$ and $y = d$. Draw a horizontal approximating rectangle.
2. At any y, determine the distance $r_{\text{in}}(y)$ from $x = L$ to the rectangle's closest vertical side. Then find the distance $r_{\text{out}}(y)$ from $x = L$ to the rectangle's farthest vertical side.
3. Use Equation (5.8) with the bounds $y = c$ and $y = d$.

The **Shell Method** is an alternative to the Disk Method and Washer Method. In the approach, we construct approximating rectangles *parallel* to the axis of revolution (instead of *perpendicular* as with the Disk Method and Washer Method). If the region bounded by $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is rotated about the y-axis, then its volume is

$$V = 2\pi \int_a^b x f(x)\, dx. \tag{5.9}$$

More generally, we have the following formulas:

- If a *vertical* approximating rectangle at x has height $h(x)$ and is a distance of $r(x)$ away from the *vertical* axis of revolution, then the solid of revolution has volume

$$V = 2\pi \int_a^b r(x)h(x)\, dx. \tag{5.10}$$

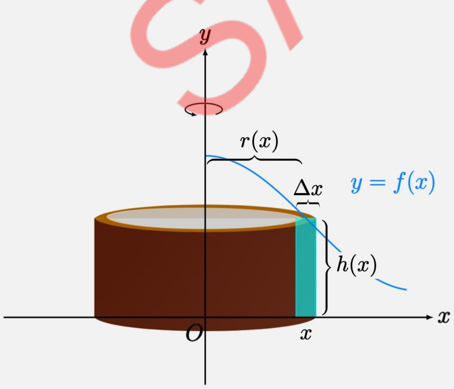


Figure 5.6

- If a *horizontal* approximating rectangle at y has width $h(y)$ and is a distance of $r(y)$ away from the *horizontal* axis of revolution, then the solid of revolution has volume

$$V = 2\pi \int_c^d r(y)h(y)\,\mathrm{d}y\,.$$

⚠ WARNING ⚠
The Disk Method, Washer Method, and Shell Method *cannot* return negative volumes.

In Figure 5.6, a shell formed by revolving a vertical approximating rectangle about the y-axis has $r(x) = x$ and $h(x) = f(x)$, so Equation (5.10) becomes Equation (5.9).

Work

A **force** is an action that tends to cause acceleration. By **Newton's Second Law**, the net force acting on an object equals the product of the object's mass and acceleration—namely,

$$F = ma = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\,,$$

where s is the position and t is time. Thus, as expected, a greater force leads to a higher acceleration.

Work measures how much energy is added to an object to make it move some distance. If a constant force F is applied over a distance d, then the work done is

$$W = Fd\,.$$

In the **International System of Units** (**SI**), the unit for force is the **newton** (**N**). The **joule** (**J**) is the product of a newton and a meter, and it is the SI unit for work. If force is measured in pounds and distance is measured in feet, then work is expressed in **foot-pounds** (**ft-lb**). Work is *independent of time*.

If a *variable* force $F(x)$ acts on an object as it is moved from $x = a$ to $x = b$, then the work done on the object is

$$W = \int_a^b F(x)\,\mathrm{d}x\,.$$

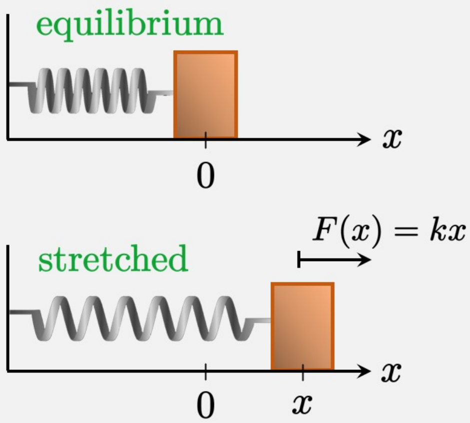


Figure 5.7

Many forces in nature are nonuniform, such as the force needed to stretch a spring.

A spring's hardness is measured by the **spring constant**, k. (A stiff spring has a large k, whereas a loose spring has a small k.) When a spring is at its natural length, it is said to be in *equilibrium*. But to stretch a spring, a force must be applied. **Hooke's law** states that to keep a spring stretched by a distance x from its equilibrium position, one must apply a force of

$$F(x) = kx\,.$$

(See Figure 5.7.) For example, if a force of 20 lb is required to stretch a spring 0.1 ft, then the spring constant is $k = F/x = 20/0.1 = 200$ lb/ft.

A gas is a substance that expands freely and has no fixed shape, such as oxygen or carbon dioxide. Consider a sample of gas enclosed by a container and sealed above by a piston. As the gas expands in volume, the piston is pushed outward—a motion that requires the gas to do work. (See Figure 5.8.) We define **pressure** to be the force per area; a gas's pressure P and volume V are related by **Boyle's Law**:

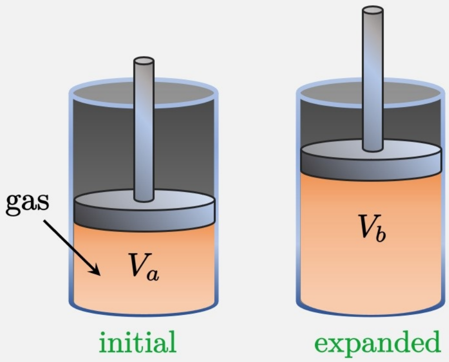


Figure 5.8

$$P = \frac{C}{V},$$

where C is a constant of proportionality. The work done by the gas as its volume increases from V_a to V_b is

$$W = \int_{V_a}^{V_b} \frac{C}{V}\,\mathrm{d}V.$$

The force of gravity acting on an object of mass m is $F_g = mg$, where g is the acceleration due to gravity, $g = 9.8\ \mathrm{m/sec}^2$. To lift an object, one must apply an upward force whose magnitude equals the gravitational force. (Doing so effectively counters the downward gravitational force.) Thus, the work needed to lift an object an upward distance h is given by $W = F_g h$, or

$$W = mgh.$$

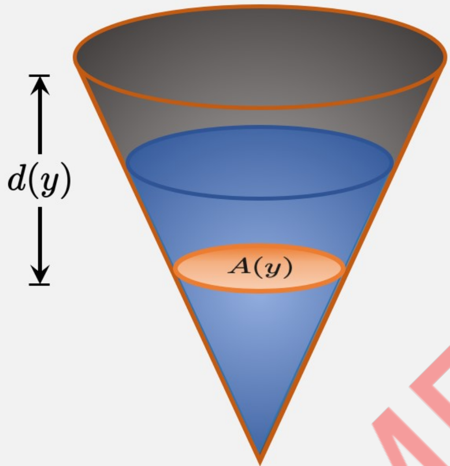


Figure 5.9

Consider the objective of pumping out all the liquid from a tank. Assume that the liquid has a uniform density of ρ (*rho*). At some level y, suppose that a strip of liquid has cross-sectional area $A(y)$ and is located a distance $d(y)$ below the top. (See Figure 5.9.) We define the **specific weight** to be $\gamma = \rho g$ (where γ is the Greek symbol *gamma*). If the liquid is between the levels $y = a$ and $y = b$, then the work done in pumping out all the liquid is given by the formula

$$W = \gamma \int_a^b d(y)A(y)\,\mathrm{d}y. \qquad (5.11)$$

For water, $\gamma = 9800\ \mathrm{N/m}^3 = 62.5\ \mathrm{lb/ft}^3$.

The following steps enable you to calculate the work done in pumping out liquid:

1. Define an axis system. For many problems, it is most convenient to establish a downward y-axis with $y = 0$ as the top of the tank.
2. Consider a thin, horizontal strip of liquid at some level y. Determine the distance $d(y)$ from the tank's top to the horizontal strip.
3. In terms of y, determine an expression for the cross-sectional area $A(y)$ of the strip. Doing so may require geometric techniques, such as analyzing similar triangles.
4. Use Equation (5.11) with a and b representing the liquid's boundaries.

TIP
For distances given in meters, use Equation (5.11) with $\gamma = 9800\ \mathrm{N/m}^3$ and report the work in joules (J). For distances given in feet, use $\gamma = 62.5\ \mathrm{lb/ft}^3$ and report the work in foot-pounds (ft-lb).

STUDYING TIPS

- Construct a sketch for any calculations involving area or volume. Graph all the curves, label any intersection points, and shade in the region of interest.
- Remember that the Disk Method is used when there is *no gap* between a region and the axis of revolution, whereas the Washer Method is used when there *is* such a gap. The Shell Method can be used in place of either.
- When using the Disk Method, Washer Method, or Shell Method, draw an approximating rectangle and label any distances to the axis of revolution.
- Remember that areas between curves and volumes of solids of revolution can *never* be negative. Check your work if you attain a negative result.
- The difficulty in this chapter's problems resides in setting up integrals. You should focus on writing correct integral setups.
- A flashcard may help you memorize the Mean Value Theorem for Integrals.
- Use geometry to visualize the Mean Value Theorem for Integrals, as in Figure 5.10.

47. The region enclosed by $y = 1/(x^2+2)$, the x-axis, the y-axis, and the line $x = 1$ is the base of a solid. At each x, the cross section to the solid perpendicular to the x-axis is a rectangle of height $3x$. Calculate the solid's volume.

48. A log has circular cross sections, and its radius r as a function of the distance x from the left end is $r(x) = 9 - x^2$ for $0 \leqslant x \leqslant 3$, where both x and $r(x)$ are measured in centimeters. Calculate the log's volume on this interval.

49. The diameter of a pipe with circular cross sections y inches above the bottom is given by $d(y) = \sqrt{50 - 3y^2}$. If the pipe is 2 inches tall, then calculate the volume of water the pipe can fit.

---

50. A baker models the base of a mini-cake using the function $y = 2 + 2\cos(\pi x/3)$ for $-3 \leqslant x \leqslant 3$, where x is measured in inches. Cross sections perpendicular to the x-axis are semicircles. For each cubic inch, the baker injects 0.1 gram of liquid chocolate into the cake. How many grams of liquid chocolate does the baker add to the cake? (Use a graphing calculator.)

51. The surface of a pond is the region enclosed by the curves

$$f(x) = 5\cos\left(\frac{\pi x}{4}\right) \qquad \text{and} \qquad g(x) = x^2,$$

where x is measured in feet. At any x, the pond's depth of water is given by the function $7 - x$. Calculate the pond's volume using a graphing calculator.

---

52. Let R be the region bounded by the parabola $y = x^2$, the x-axis, the y-axis, and the line $x = k$, where k is a positive constant. Let S be the solid whose base is region R and whose cross sections perpendicular to the x-axis are squares. Find the value of k for which the area of R and the volume of S have equal magnitudes.

53. Consider the family of functions

$$f(x) = \frac{1}{(kx)^2 + 1},$$

where k is a positive constant. The region under the graph of $y = f(x)$ and above the x-axis from $x = 1$ to $x = 2$ is the base of a solid whose cross sections perpendicular to the x-axis are rectangles of uniform height k. For what value of k is the solid's volume maximized?

Solids of Revolution

54. (a) When do we use the Disk Method, and when do we use the Washer Method?

(b) Fill in the blanks: Using the Shell Method, when we rotate a region about a vertical line, we consider ______ approximating rectangles and integrate with respect to x. But when we rotate a region about a horizontal line and use the Shell Method, we consider ______ approximating rectangles and integrate with respect to y.

55–76: Calculate the volume of the solid generated by revolving the bounded region around the specified axis.

55. $y = \sqrt{x},\ y = 0,\ x = 16;$ about x-axis

56. $y = x - 4,\ x = 0,\ x = 2,\ y = 0;$ about x-axis

57. $y = e^{-2x},\ x = 0,\ x = e,\ y = 0;$ about x-axis

58. $y = x^2,\ y = 0,\ y = 1,\ x = -1,\ x = 1;$ about $y = 1$

59. $y = \frac{1}{x},\ x = 1,\ x = 3,\ y = -1;$ about $y = -1$

60. $y = x^2,\ x = 0,\ y = 0,\ y = 8;$ about y-axis

61. $x = e^y,\ x = 0,\ y = 1,\ y = 2;$ about y-axis

62. $y = e^x,\ y = 0,\ x = 0,\ x = 1;$ about $y = -1$

63. $y = x^2,\ y = x,\ x = 0,\ y = 0;$ about $y = 1$

64. $y = -\sqrt{4 - x},\ x = 4 - y,\ y = 0;$ about y-axis

65. $y = x^2,\ y = x,\ x = 0;$ about x-axis

66. $x = -y^2,\ x = -1;$ about $x = -1$

67. $y = e^x,\ y = e^{-x},\ y = 0,\ x = -1,\ x = 1;$ about x-axis

68. $y = x^2,\ y = 4 - x^2;$ about $y = -1$

69. $y = \sqrt{x - 1},\ x = 0,\ y = 0,\ y = 4;$ about $x = -3$

70. $y = x^2,\ y = 3 - 2x,\ y = 0;$ about x-axis

71. $y = \sqrt{x},\ x = \sqrt{y},\ x = 1;$ about $x = -1$

72. $x = 5 - y,\ y = x + 1,\ y = 1;$ about $x = -2$

73. $y = 2 - x,\ y = \sqrt{x},\ y = 0;$ about x-axis

74. $y = 1 - x^2,\ y = -2;$ about $y = 3$

75. $y = 1 + x^2,\ x = 0,\ x = 2,\ y = 0;$ about y-axis

76. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{4};$ about $y = -1$

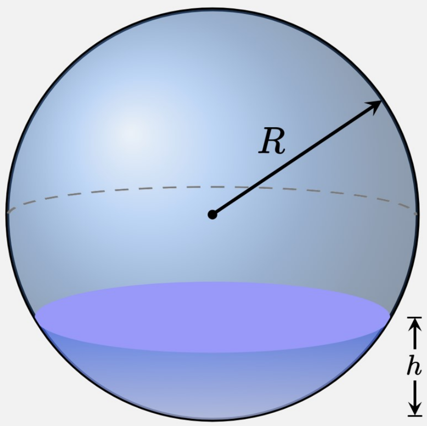
---

77. Using the Disk Method, prove that a right cone of base h and radius r has volume

$$V = \tfrac{1}{3}\pi r^2 h.$$

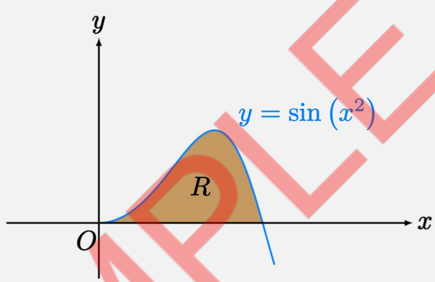
78. The edge of a bowl is a parabola. The top of the bowl is a circle of diameter 8 inches, and its height is 6 inches. Soup is poured into the bowl to reach a liquid level of 2 inches above the base. Calculate the volume of soup added.

79. In a sphere of radius R, find the volume of the bottom cap of height h.



80. Let R be the region bounded by $y = \sin\left(x^2\right)$ and the x-axis. Write an integral expression that gives the volume of the solid generated by rotating R

(a) about the y-axis
(b) about the x-axis



81–97: Use the Shell Method to find the volume of the solid generated by rotating the bounded region about the specified line.

81. $y = 2x(2-x)$, $x = 0$; about y-axis

82. $y = x$, $y = x^2$; about y-axis

83. $y = x - x^3$, $x = 0$, $y = 0$; about y-axis

84. $y = 1 + \sqrt{x}$, $y = 0$, $x = 1$, $x = 4$; about y-axis

85. $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$, $x = 2$; about y-axis

86. $y = \sqrt{x}$, $y = 0$, $x = 4$; about x-axis

87. $y = x$, $y = x^2$; about $x = 4$

88. $x = 4\sqrt{y}$, $x = 0$, $y = 1$; about $y = 4$

89. $y = 4x$, $y = 2\sqrt{x}$; about x-axis

90. $y = 2$, $y = 1 + x^2$, $x = 0$; about $x = -\frac{1}{2}$

91. $x = 3$, $x = y^2 - 1$; about $y = -5$

92. $y = x^2$, $y = 8 - x^2$; about $x = 6$

93. $y = \sqrt{x}$, $y = 2 - x$, $y = 0$; about $y = -1$

94. $y = x + 3$, $y = 6 - 2x$, $y = -1$; about $y = -7$

95. $y = e^{-x^2/2}$, $y = 0$, $x = 0$, $x = 4$; about y-axis

96. $x = \sqrt{y+6}$, $y = x$, $y = -5$; about $y = -6$

97. $y = \dfrac{2}{x^2+1}$, $y = 1$, $x = 0$; about y-axis

98–101: The integral equals the volume of a solid of revolution. Identify the axis of revolution and describe the revolved region.

98. $2\pi \displaystyle\int_0^{\pi/2} x\cos x \, dx$

99. $2\pi \displaystyle\int_0^{4} x^4 \, dx$

100. $2\pi \displaystyle\int_{-2}^{1} (7-x)e^x \, dx$

101. $2\pi \displaystyle\int_0^{0.2} (y+3)\left(2y^2 - \sqrt{y}\right) dy$

102–105: Use a graphing calculator with the Shell Method to find the volume of the solid generated by rotating the region about the specified line.

102. $y = \cos x$, $y = -\cos x$, $x = 0$; about $x = -2$

103. $y = \sin^{-1} x$, $x = 0$, $y = -\dfrac{\pi}{4}$; about $x = 1$

104. $x = e^y$, $x = 2 + e^{3y}$, $y = -3$, $y = 1$; about $y = 3$

105. $y = e^{2x}$, $y = 4 + \sqrt{x}$, $x = 0$, $y = 1$, $y = 4.3$; about $y = -0.4$

Work

106. In each case, calculate the work done.

(a) A boy applies a force of 20 newtons to push a toy 0.4 meter
(b) A car is driven a distance of 100 feet as the engine supplies a force of 7000 pounds
(c) A trainer lifts a 10-kilogram dumbbell 1.2 meters above his waist
(d) A dropped 3-pound water bottle is placed back on a table of height 5 feet

Chapter 5 Exercise Solutions

1. (a) We note that $x \geqslant x^2$ for all x in $[0,1]$. Accordingly, we integrate the top function x minus the bottom function x^2 from $x=0$ to $x=1$:

$$A = \int_0^1 (x - x^2)\,\mathrm{d}x = \left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 = \boxed{\tfrac{1}{6}}$$

(b) To integrate with y, we need to ensure that R is bounded by the same two functions over the entire region. The right boundary is $y = x^2$ and the left boundary is x; these two functions cover the entire region R. We need both functions in terms of y, so we write $y = x^2$ as $x = \sqrt{y}$. Then we integrate from $y = 0$ to $y = 1$ (integrating the right function minus the left function):

$$A = \int_0^1 (\sqrt{y} - y)\,\mathrm{d}y$$

$$= \left(\tfrac{2}{3}y^{3/2} - \tfrac{1}{2}y^2\right)\Big|_0^1$$

$$= \tfrac{2}{3} - \tfrac{1}{2} = \boxed{\tfrac{1}{6}}$$

(c) The answers in both parts are the same. This shows that we have the freedom to choose to integrate using either x or y, as we should obtain the same result. But sometimes one method is easier than another, so it is important to remain flexible.

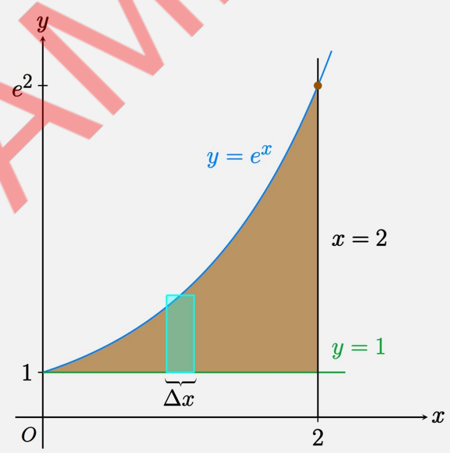
2. It is imperative to sketch the bounded area. For $0 \leqslant x \leqslant 2$, the graph of $y = e^x$ is above the graph of $y = 1$. The bounded region is between the y-axis (or $x = 0$) and $x = 2$, so these are the integral bounds. Thus, the integral setup is

$$A = \int_0^2 (e^x - 1)\,\mathrm{d}x$$

$$= (e^x - x)\Big|_0^2$$

$$= (e^2 - 2) - (e^0 - 0)$$

$$= \boxed{e^2 - 3} \approx 4.389\,.$$



3. We first determine the intersection points by equating the two functions:

$$x = \sqrt{x} \implies x = 0, 1\,.$$

The graph of $y = \sqrt{x}$ is above the graph of $y = x$ over $0 \leqslant x \leqslant 1$, so we integrate with x, subtracting the lower function from the upper function:

$$A = \int_0^1 (\sqrt{x} - x)\,\mathrm{d}x$$

$$= \left(\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2\right)\Big|_0^1$$

$$= \left(\tfrac{2}{3} - \tfrac{1}{2}\right) - 0$$

$$= \boxed{\tfrac{1}{6}}$$

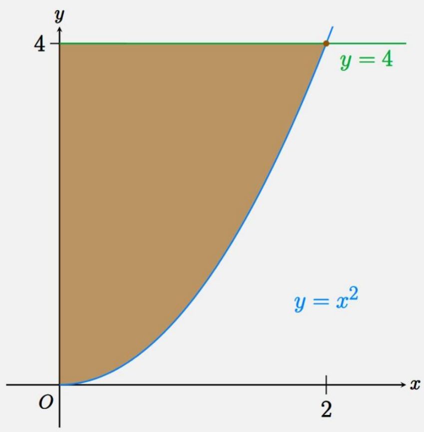
4. This region is in the first quadrant, bounded above by the line $y = 4$ and below by the parabola $y = x^2$. The two functions intersect at $(2, 4)$. It is easy to integrate either with x or y. If we integrate with x, then the integrand is the top function ($y = 4$) minus the bottom function ($y = x^2$) from $x = 0$ to $x = 2$—namely,

$$A = \int_0^2 (4 - x^2)\,\mathrm{d}x\,.$$

With y, we rewrite $y = x^2$ as $x = \sqrt{y}$. We integrate the right-bounding function $x = \sqrt{y}$ minus the left-bounding function, the line $x = 0$. This gives

$$A = \int_0^4 (\sqrt{y} - 0)\,\mathrm{d}y\,.$$

Either setup confers the answer of $\boxed{16/3}$.



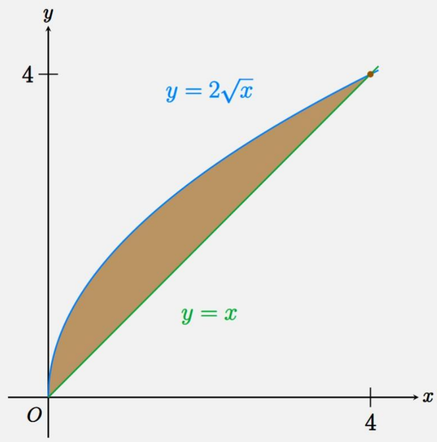
5. This region is in the first quadrant, bounded above by $y = 2\sqrt{x}$ and below by the line $y = x$. The two functions intersect at $(4, 4)$ if we solve $2\sqrt{x} = x$. Integrals with x or y are easy to write. If we integrate with x, then the integrand is the top function ($y = 2\sqrt{x}$) minus the bottom function ($y = x$) from $x = 0$ to $x = 4$—that is,

$$A = \int_0^4 (2\sqrt{x} - x)\,\mathrm{d}x\,.$$

If we integrate with y, then we rewrite $y = 2\sqrt{x}$ as $x = (y/2)^2$. We integrate, from $y = 0$ to $y = 4$, the right-bounding function $x = y$ minus the left-bounding function $x = (y/2)^2$. This gives

$$A = \int_0^4 \left[y - \left(\tfrac{y}{2}\right)^2\right]\mathrm{d}y\,.$$

Either method gives the answer of $\boxed{8/3}$.



6. The bounded region is in the first and second quadrants, with an upper boundary of $y = 2\cos x$ and lower boundary of $y = 1$. The two functions intersect at the points $(-\pi/3, 1)$ and $(\pi/3, 1)$. If we choose to integrate with x, then our integral is

$$A = \int_{-\pi/3}^{\pi/3} (2\cos x - 1)\,dx.$$

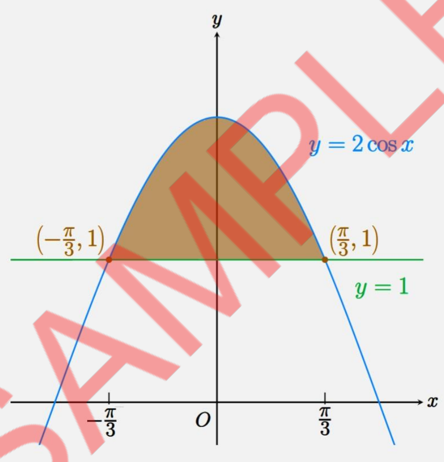
Or we could realize that the region is symmetric about the y-axis, so the area is twice the area of one region:

$$A = 2\int_{0}^{\pi/3} (2\cos x - 1)\,dx.$$

Evaluating the integral gives

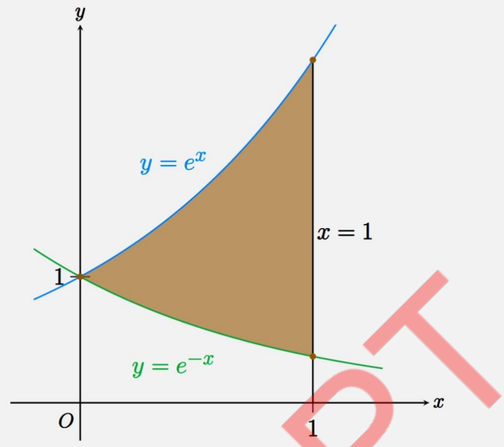
$$2(2\sin x - x)\Big|_0^{\pi/3} = \boxed{2\sqrt{3} - \tfrac{2}{3}\pi} \approx 1.368.$$

It is important to recognize symmetries because they permit easier problem-solving. Choosing to integrate with y is very difficult, so we omit that setup.



7. We observe that the area is bounded above by $y = e^x$, below by $y = e^{-x}$, and right by $x = 1$. It is easiest to integrate with x because the upper and lower boundaries do not change anywhere in the region. The region goes from $x = 0$ to $x = 1$, so the area is

$$\begin{aligned} A &= \int_0^1 \left(e^x - e^{-x}\right)\,dx \\ &= \left(e^x + e^{-x}\right)\Big|_0^1 \\ &= \boxed{e + \tfrac{1}{e} - 2} \approx 1.086. \end{aligned}$$

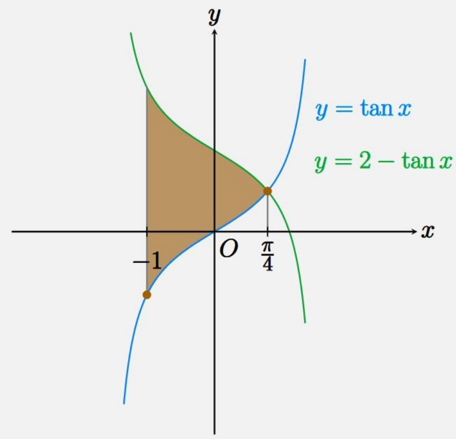


8. Since $2 - \tan x \geqslant \tan x$ on $[-1, 1]$, the region is bounded above by $y = 2 - \tan x$ and below by $y = \tan x$. Both curves intersect when

$$2 - \tan x = \tan x \implies \tan x = 1 \implies x = \frac{\pi}{4}.$$

Integrating with x, we find the area to be

$$\begin{aligned} A &= \int_{-1}^{\pi/4} [(2 - \tan x) - \tan x]\,dx \\ &= \int_{-1}^{\pi/4} (2 - 2\tan x)\,dx \\ &= (2x - 2\ln|\sec x|)\Big|_{-1}^{\pi/4} \\ &= 2\left[\frac{\pi}{4} - (-1)\right] - 2\left[\ln\left(\sec\frac{\pi}{4}\right) - \ln(\sec(-1))\right] \\ &= \boxed{\frac{\pi}{2} + 2 - 2\ln\left[\cos(-1)\sqrt{2}\right]} \approx 4.109. \end{aligned}$$



9. The graphs intersect at $(3, -1)$ and $(3, 3)$. The right boundary is $x = 3$, and the left boundary is $x = (y - 1)^2 - 1$. It is easiest to integrate with y. Because the region spans from $y = -1$ to $y = 3$, its

26. We find the solid's volume by integrating $A(x)$ from $x=0$ to $x=3$, as follows:

$$V = \int_0^3 A(x)\,\mathrm{d}x = \int_0^3 (x^2+2)\,\mathrm{d}x$$

$$= \left(\tfrac{1}{3}x^3 + 2x\right)\Big|_0^3 = \boxed{15}$$

27. The solid's volume is given by integrating $A(y)$ from $y=4$ to $y=9$, as follows:

$$V = \int_4^9 A(y)\,\mathrm{d}y = \int_4^9 (\sqrt{y}+3)\,\mathrm{d}y$$

$$= \left(\tfrac{2}{3}y^{3/2} + 3y\right)\Big|_4^9$$

$$= 45 - \tfrac{52}{3} = \boxed{\tfrac{83}{3}}$$

28. Let's think of the xy-plane as the "ground." Because cross sections *perpendicular* to the x-axis are squares, the cross-sectional area to the solid at any x is $A(x) = \left(4-x^2\right)^2$. An approximating slice of the solid is

$$\Delta V = A(x)\Delta x = \left(4-x^2\right)^2 \Delta x.$$

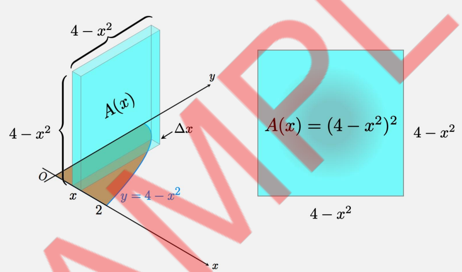
Hence, the solid's volume V is given by integrating $A(x)$ from $x=0$ to $x=2$, as follows:

$$V = \int_0^2 A(x)\,\mathrm{d}x = \int_0^2 \left(4-x^2\right)^2\,\mathrm{d}x$$

$$= \int_0^2 \left(16 - 8x^2 + x^4\right)\mathrm{d}x$$

$$= \left(16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_0^2$$

$$= \boxed{\tfrac{256}{15}} \approx 17.067.$$



29. Let's consider a single square within $1 \leqslant x \leqslant 2$: The side of this square is x^2, meaning its cross-sectional area is given by

$$A(x) = \left(x^2\right)^2 = x^4.$$

Accordingly, we find the solid's volume by integrating this function from $x=1$ to $x=2$—namely,

$$V = \int_1^2 A(x)\,\mathrm{d}x = \int_1^2 x^4\,\mathrm{d}x$$

$$= \tfrac{1}{5}x^5\Big|_1^2$$

$$= \tfrac{1}{5}\left(2^5 - 1^5\right) = \boxed{\tfrac{31}{5}}$$

30. At each x in $[-2,1]$, the cross-sectional area to the solid perpendicular to the x-axis is

$$A(x) = (x+2)^2.$$

Accordingly, the solid's total volume is given by integrating this function from $x=-2$ to $x=1$:

$$V = \int_{-2}^1 A(x)\,\mathrm{d}x = \int_{-2}^1 (x+2)^2\,\mathrm{d}x$$

$$= \tfrac{1}{3}(x+2)^3\Big|_{-2}^1$$

$$= \tfrac{1}{3}(27-0) = \boxed{9}$$

31. At each x, the cross section to the solid is a rectangle of dimensions $y = \sqrt{x}$ and $2y = 2\sqrt{x}$. Thus, at each x the cross-sectional area to the solid is

$$A(x) = \left(\sqrt{x}\right)\left(2\sqrt{x}\right) = 2x.$$

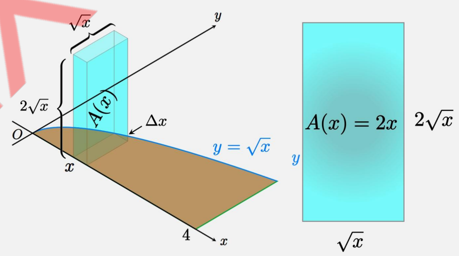
An approximating rectangular prism has volume

$$\Delta V = A(x)\Delta x = 2x\Delta x.$$

The volume of S is therefore given by integrating $A(x)$ from $x=0$ to $x=4$:

$$V = \int_0^4 A(x)\,\mathrm{d}x = \int_0^4 2x\,\mathrm{d}x$$

$$= x^2\Big|_0^4 = \boxed{16}$$



32. At each x, the cross section to the solid is a right isosceles triangle whose hypotenuse has length $y = \sqrt{9-x^2}$. Accordingly, the triangle's legs each have length $y/\sqrt{2}$. So at each x, the cross-sectional area of the solid is

$$A(x) = \frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2 = \frac{y^2}{4} = \frac{9-x^2}{4}.$$

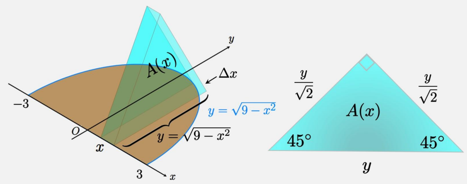
Then the solid's volume is given by integrating $A(x)$ from $x=-3$ to $x=3$:

$$V = \int_{-3}^3 A(x)\,\mathrm{d}x = \int_{-3}^3 \frac{9-x^2}{4}\,\mathrm{d}x.$$

But using symmetry (since the integrand is even), we simplify the calculation as follows:

$$V = 2\cdot\int_0^3 \frac{9-x^2}{4}\,\mathrm{d}x = \tfrac{1}{2}\int_0^3 (9-x^2)\,\mathrm{d}x$$

$$= \tfrac{1}{2}\left(9x - \tfrac{1}{3}x^3\right)\Big|_0^3 = \boxed{9}$$



geometric power series, 420
global
  extrema, *see* absolute extrema
  maximum, *see* absolute maximum
  minimum, *see* absolute minimum
growth constant, 318

H

half-life, 318
Harmonic series, 413
  alternating version, 416
Hooke's law, 172
horizontal asymptote, 10
horizontal tangent, 45
hydrostatic force, 268
hydrostatic pressure, 268

I

Implicit Differentiation, 45
implicit equation, 45
improper integral, 222
  Type I, 222
  Type II, 223
increasing
  for a function, 84
  for a sequence, 412
indeterminate form
  $0 \times \infty$, 86
  0^0, 86
  1^∞, 86
  $\frac{0}{0}$, 5
  $\infty - \infty$, 86
  ∞/∞, 10
  ∞^0, 86
infinite discontinuity, 9
infinite geometric series, 413
infinite sequence, 411
infinite series, 412
inflection point, 84
initial condition, 130, 315
initial value problem, 130, 315
integrable, 131
integral sign, 129
integral(s)
  by substitution, *see* Substitution Rule
  definite, 131
  indefinite, 129
  of powers of sine or cosine, 215
  of symmetric functions, 134
  of trigonometric functions, 129
  with cosecant and cotangent, 218
  with parametric functions, 366
  with secant and tangent, 217
  with sine and cosine, 216
integrand, 129
integrating factor, 321
Integration by Parts, 215
Intermediate Value Theorem, 9
International System of Units, 172
interval of convergence, 419

J

joule (unit), 172
jump discontinuity, 8

L

L'Hôpital's Rule, 85
Lagrange Error Bound, *see* Taylor's Remainder Theorem
lamina, 267
Law of Demand, 87
Law of Supply, 266
left-endpoint approximation, 137
Leibniz notation, 41
lemniscate (polar graph), 368
limacon (polar graph), 368
Limit Comparison Test, 415
limit definition of a derivative, 41
limit laws, 4
  for sequences, 411
limit(s), 3
  at infinity, 4
  by change of variable, 8
  infinite, 4
  of a sequence, 411
  of trigonometric functions, 7
limits of integration, *see* bounds of integration
line
  normal, *see* normal line
  polar equations of, 368
  secant, *see* secant line
  tangent, *see* tangent line
linearization, 47
local extrema, *see* relative extrema
logarithm laws, 46
Logarithmic Differentiation, 46
logistic
  decay, 320
  growth, 320
  properties of, 320

M

Maclaurin polynomial, 422