ValsCalc

## Chapter 4 Free-Response Practice Test

**Directions**: This practice test features free-response questions based on the content in Chapter 4: Integration.

**4.1**: Antidifferentiation
**4.2**: Definite Integrals
**4.3**: Fundamental Theorem of Calculus
**4.4**: Integration by Substitution
**4.5**: Numerical Integration

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

*Visit valscalc.com to access a free online calculus textbook and additional multiple-choice and free-response practice exams.*

**Integration**

**Number of Questions—14**

**Suggested Time—1 hour 30 minutes**

**NO CALCULATOR**

**Scoring Chart**

| Section | Points Earned | Points Available |
|---|---|---|
| **Short Questions** | | 55 |
| **Question 12** | | 15 |
| **Question 13** | | 15 |
| **Question 14** | | 15 |
| **TOTAL** | | **100** |

## Short Questions

**1.** Evaluate $\int \left(2x^3 - e^x - \csc x \cot x\right)\,\mathrm{d}x$. *(5 pts.)*

**2.** Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{2x-4} \frac{\cos t}{\sqrt{t}}\,\mathrm{d}t$. *(5 pts.)*

**3.** Calculate the area between the graph of $y = 3\sec x\tan x$ and the $x$-axis from $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{3}$. *(5 pts.)*

**4.** Solve the initial value problem $y' = \cos\theta + 2\sin\theta$, $y\left(\dfrac{\pi}{2}\right) = 2$. *(5 pts.)*

**5.** For a particle's acceleration function $a(t) = 8t^3 - e^t + 2$, $t \geqslant 0$, find the position function given a position of 2 and velocity of 4 when $t = 1$. *(5 pts.)*

**6.** Given $-e^{x/2} \leqslant e^{x/2} \sin\left(\sqrt[5]{x}\right) \leqslant e^{x/2}$ for all $x$, bound $\int_0^2 e^{x/2} \sin\left(\sqrt[5]{x}\right)\, \mathrm{d}x$ between two numbers. *(5 pts.)*

**7.** Find $\displaystyle\int \frac{6x}{e^{3x^2+4}}\,\mathrm{d}x.$ *(5 pts.)*

**8.** Calculate $\displaystyle\int_1^{e^{\pi/6}} \frac{4}{x\cos(2\ln x)}\,\mathrm{d}x.$ *(5 pts.)*

**9.** Water flows into a tank at a rate given by $r(t)$, where $r(t)$ is measured in liters per minute and $t \geqslant 0$ is time measured in minutes. The following table shows selected values of $r(t)$. Use the Trapezoidal Rule to approximate the volume of water that enters the tank between $t = 0$ and $t = 8$. *(5 pts.)*

| $t$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $r(t)$ | 1 | 4 | 3 | 0 | 6 |

**10.** Use the limit of a Riemann sum to calculate $\int_1^4 2x\,\mathrm{d}x$. *(5 pts.)*

**11.** Determine $\int \frac{1}{3x^2+12x+24}\,\mathrm{d}x$. *(5 pts.)*

## Long Questions

**12.** A cannon on a platform 80 feet above the ground launches a projectile straight upward with an initial speed of 64 feet per second. The projectile continues upward before later falling back to the ground. The acceleration due to gravity is 32 feet per second squared. Consider the upward direction to be positive.

(a) Show that the projectile's velocity function, in feet per second, is $v(t) = -32t + 64$ for $t \geqslant 0$ measured in seconds. *(3 pts.)*

(b) Show that the projectile's height as a function of time is modeled by $h(t) = -16t^2 + 64t + 80$. *(4 pts.)*

(c) When does the projectile reach the ground? *(2 pts.)*

(d) Calculate the projectile's maximum height. *(3 pts.)*

(e) Calculate the projectile's speed immediately before it strikes the ground. *(3 pts.)*

**13.** This question explores several techniques for approximating the value of the definite integral $\int_1^5 e^{-2x}\,\mathrm{d}x$. In each part, report an exact answer in terms of $e$.

(a) Use the Midpoint Rule with $n = 2$ to estimate $\int_1^5 e^{-2x}\,\mathrm{d}x$. (*3 pts.*)

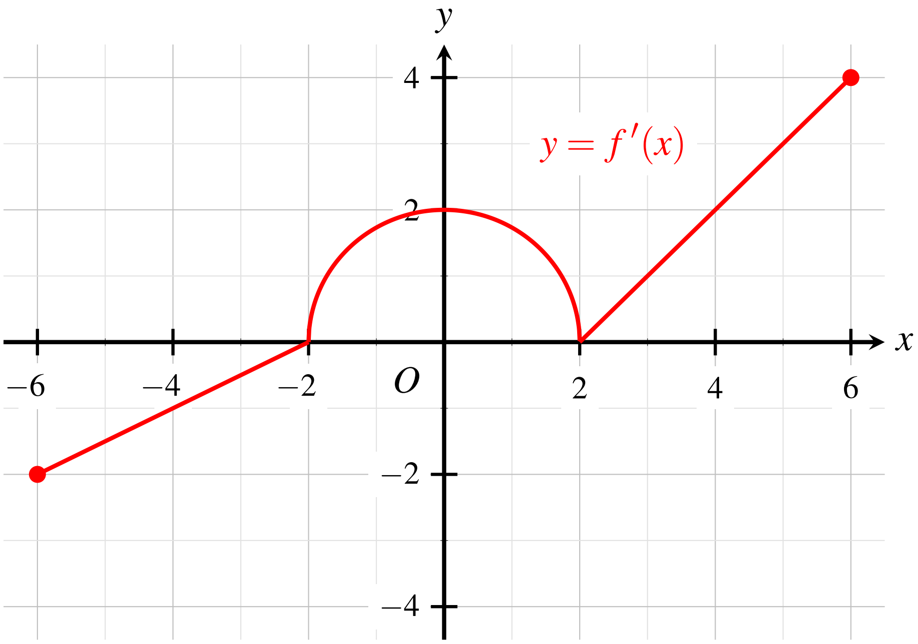
(b) Estimate $\int_1^5 e^{-2x}\,\mathrm{d}x$ using a left-endpoint approximation and a right-endpoint approximation, each with $n = 4$. (*4 pts.*)

(c) Use Simpson's Rule with $n = 4$ to approximate $\int_1^5 e^{-2x}\,\mathrm{d}x$. (*3 pts.*)

(d) Construct an error bound to the approximation in part (c). *(5 pts.)*

**14.** The following figure shows the graph of $y = f'(x)$, the derivative of $f$, on $-6 \leqslant x \leqslant 6$.

The graph consists of a semicircle on $-2 < x < 2$ and linear portions on $-6 < x < -2$ and $2 < x < 6$. The function $f$ satisfies $f(-2) = 5$. Another function $g$ is defined by $g(x) = \int_{-2}^{x} f'(t)\,\mathrm{d}t$.




(a) Calculate $g(2)$, $g(-2)$, and $g(-6)$. (3 pts.)

(b) Determine $f(4)$. *(2 pts.)*

(c) Find the $x$-coordinates of any inflection points of the graph of $y = g(x)$. Justify your answers. *(3 pts.)*

(d) Given that $f''$ is continuous on $[3,6]$, calculate $\int_3^6 \left[2f''(x) - 4\right] \, dx$. *(3 pts.)*

(e) Obtain an expression for $f(x)$ for $-6 < x < -2$. *(4 pts.)*

*This marks the end of the test. The solutions and scoring rubric begin on the next page.*

## Short Questions (5 points each)

**1.** The antiderivatives of each term are as follows:

$$\int 2x^3\,\mathrm{d}x = \frac{2x^4}{4} + C = \frac{x^4}{2} + C, \quad *$$

$$\int e^x\,\mathrm{d}x = e^x + C, \quad *$$

$$\int \csc x\cot x\,\mathrm{d}x = -\csc x + C. \quad *$$

Thus,

$$\int (2x^3 - e^x - \csc x\cot x)\,\mathrm{d}x = \int 2x^3\,\mathrm{d}x - \int e^x\,\mathrm{d}x - \int \csc x\cot x\,\mathrm{d}x \quad *$$

$$= \boxed{\frac{x^4}{2} - e^x + \csc x + C} \quad *$$

**2.** By Part I of the Fundamental Theorem of Calculus with the Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{2x-4} \frac{\cos t}{\sqrt{t}}\,\mathrm{d}t = \frac{\cos(2x-4)}{\sqrt{2x-4}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2x-4) \quad **$$

$$= \boxed{\frac{2\cos(2x-4)}{\sqrt{2x-4}}} \quad ***$$

**3.** Because $\sec x\tan x \geqslant 0$ on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$, the area is

$$\int_{\pi/6}^{\pi/3} 3\sec x\tan x\,\mathrm{d}x = 3\sec x\Big|_{\pi/6}^{\pi/3} \quad ***$$

$$= 3\sec\left(\frac{\pi}{3}\right) - 3\sec\left(\frac{\pi}{6}\right) \quad *$$

$$= 3(2) - 3\left(\frac{2}{\sqrt{3}}\right)$$

$$= \boxed{6 - 2\sqrt{3}} \quad *$$

**4.** Antidifferentiating $y' = \cos\theta + 2\sin\theta$ gives

$$y = \sin\theta - 2\cos\theta + C$$ **

for some constant $C$. Substituting the initial condition $y\left(\frac{\pi}{2}\right) = 2$ yields

$$y\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{\pi}{2}\right) + C = 2$$ *

$$1 + C = 2$$

$$\implies C = 1\,.$$ *

Thus, the solution is

$$\boxed{y = \sin\theta - 2\cos\theta + 1}$$ *

**5.** Because the velocity function is $v(t) = \int a(t)\,\mathrm{d}t$, antidifferentiating the acceleration function gives

$$v(t) = 2t^4 - e^t + 2t + C$$ *

for some constant $C$. Substituting the initial condition $v(1) = 4$ shows

$$v(1) = 2(1)^4 - e^1 + 2(1) + C = 4$$

$$\implies C = e\,.$$

Thus, the velocity function is

$$v(t) = 2t^4 - e^t + 2t + e\,.$$ *

The position function is $s(t) = \int v(t)\,\mathrm{d}t$, so antidifferentiating $v(t)$ gives

$$s(t) = \tfrac{2}{5}t^5 - e^t + t^2 + et + D\,,$$ *

where $D$ is a constant. Substituting the initial condition $s(1) = 2$ yields

$$s(1) = \tfrac{2}{5}(1)^5 - e^1 + (1)^2 + e(1) + D = 2$$

$$\implies D = \tfrac{3}{5}\,.$$ *

Accordingly, the position function is

$$s(t) = \boxed{\tfrac{2}{5}t^5 - e^t + t^2 + et + \tfrac{3}{5}}$$ *

**6.** Inequalities are conserved within definite integrals, so

$$-\int_0^2 e^{x/2}\,\mathrm{d}x \leqslant \int_0^2 e^{x/2}\sin\left(\sqrt[5]{x}\right)\,\mathrm{d}x \leqslant \int_0^2 e^{x/2}\,\mathrm{d}x\,.$$ *

To evaluate $\int_0^2 e^{x/2}\,\mathrm{d}x$, we substitute

$$u = \frac{x}{2} \implies \mathrm{d}u = \frac{\mathrm{d}x}{2}\,.$$ *

Then $\mathrm{d}x = 2\mathrm{d}u$. When $x = 0$, $u = \frac{0}{2} = 0$; when $x = 2$, $u = \frac{2}{2} = 1$. Thus,

$$\int_0^2 e^{x/2}\,\mathrm{d}x = 2\int_0^1 e^u\,\mathrm{d}u$$ *

$$= 2e^u\Big|_0^1$$ *

$$= 2(e^1 - e^0)$$

$$= 2e - 2\,.$$

Hence,

$$-(2e-2) \leqslant \int_0^2 e^{x/2}\sin\left(\sqrt[5]{x}\right)\,\mathrm{d}x \leqslant 2e - 2$$

$$\implies \boxed{2 - 2e \leqslant \int_0^2 e^{x/2}\sin\left(\sqrt[5]{x}\right)\,\mathrm{d}x \leqslant 2e - 2}$$ *

**7.** We substitute

$$u = 3x^2 + 4 \implies \mathrm{d}u = 6x\,\mathrm{d}x\,.$$ *

Then $\mathrm{d}x = \dfrac{\mathrm{d}u}{6x}$. Accordingly,

$$\int \frac{6x}{e^{3x^2+4}}\,\mathrm{d}x = \int \frac{1}{e^u}\,\mathrm{d}u$$ *

$$= \int e^{-u}\,\mathrm{d}u\,.$$

To evaluate $\int e^{-u}\,\mathrm{d}u$, we substitute

$$v = -u \implies \mathrm{d}v = -\mathrm{d}u\,.$$ *

Thus, the integral becomes

$$-\int e^{v}\,\mathrm{d}v = -e^{v} + C$$ *

$$= -e^{-u} + C$$

$$= \boxed{-e^{-3x^2-4} + C}$$ *

**8.** We substitute

$$u = 2\ln x \implies \mathrm{d}u = \frac{2}{x}\,\mathrm{d}x\,.$$ *

It then follows that $\mathrm{d}x = \dfrac{x}{2}\,\mathrm{d}u$. When $x = 1$, $u = 2\ln 1 = 0$; when $x = e^{\pi/6}$, $u = 2\ln(e^{\pi/6}) = \dfrac{\pi}{3}$. Hence,

$$\int_1^{e^{\pi/6}} \frac{4}{x\cos(2\ln x)}\,\mathrm{d}x = \int_0^{\pi/3} 2\sec u\,\mathrm{d}u$$ *

$$= 2\ln|\sec u + \tan u|\Big|_0^{\pi/3}$$ *

$$= 2\ln\left|\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right| - 2\ln|\sec 0 + \tan 0|$$ *

$$= 2\ln\left(2+\sqrt{3}\right) - 2\ln 1$$

$$= \boxed{2\ln(2+\sqrt{3})}$$ *

**9.** The amount of water that enters the tank from $t = 0$ to $t = 8$ is the accumulation of the rate on $[0, 8]$, that is, $\int_0^8 r(t)\,\mathrm{d}t$. The subintervals are $[0,2],[2,4],[4,6]$, and $[6,8]$, so we have

$$n = 4 \qquad \text{and} \qquad \Delta t = \frac{8-0}{4} = 2\,.$$ *

Accordingly, by the Trapezoidal Rule,

$$\int_0^8 r(t)\,\mathrm{d}t \approx \frac{\Delta t}{2}\left[r(0)+2r(2)+2r(4)+2r(6)+r(8)\right]$$ *

$$= \frac{2}{2}\left[1+2(4)+2(3)+2(0)+6\right]$$ **

$$= \boxed{21\,\mathrm{L}}$$ *

**10.** With $a = 1$ and $b = 4$, we have

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}\,.$$

In addition, $f(x) = 2x$. It is most convenient to use a right-hand Riemann sum; then

$$f(x_i) = f(a+i\Delta x) = f\left(1+\frac{3i}{n}\right) = 2\left(1+\frac{3i}{n}\right)\,.$$ *

Thus,

$$\begin{aligned}
\int_1^4 2x\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x & * \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} 2\left(1+\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\left(1+\frac{3i}{n}\right) \\
&= \lim_{n\to\infty}\frac{6}{n}\left[\sum_{i=1}^{n}(1)+\frac{3}{n}\sum_{i=1}^{n} i\right] & * \\
&= \lim_{n\to\infty}\frac{6}{n}\left[n+\frac{3}{n}\cdot\frac{n(n+1)}{2}\right] & * \\
&= \lim_{n\to\infty}\left[6+\frac{9(n+1)}{n}\right] \\
&= 6+9 \\
&= \boxed{15} & *
\end{aligned}$$

**11.** Factoring the denominator and completing the square, we have

$$\begin{aligned}
\frac{1}{3}\int\frac{1}{x^2+4x+8}\,\mathrm{d}x &= \frac{1}{3}\int\frac{1}{(x+2)^2+4}\,\mathrm{d}x & * \\
&= \frac{1}{12}\int\frac{1}{\frac{(x+2)^2}{4}+1}\,\mathrm{d}x \\
&= \frac{1}{12}\int\frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,\mathrm{d}x. & *
\end{aligned}$$

We substitute

$$u=\frac{x+2}{2} \implies \mathrm{d}u=\frac{\mathrm{d}x}{2}.$$

Then $\mathrm{d}x = 2\,\mathrm{d}u$. Accordingly,

$$\frac{1}{12}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,\mathrm{d}x = \frac{1}{6}\int \frac{1}{u^2+1}\,\mathrm{d}u \quad *$$

$$= \frac{1}{6}\tan^{-1} u + C \quad *$$

$$= \boxed{\frac{1}{6}\tan^{-1}\left(\frac{x+2}{2}\right)+C} \quad *$$

## Long Questions (15 points each)

**12.** (a) The acceleration function is

$$a(t) = -32 \quad *$$

(negative because gravity acts downward and upward is the positive direction). Antidifferentiating this result gives

$$v(t) = -32t + C \quad *$$

for some constant $C$. The initial velocity is 64 ft/sec, so substituting this condition shows

$$v(0) = -32(0) + C = 64$$
$$\implies C = 64\,.$$

Hence, the velocity function is

$$v(t) = \boxed{-32t + 64} \quad *$$

(b) The height $h(t)$ is a position function, so $h(t) = \int v(t)\,dt$. Antidifferentiating the result from part (a) gives

$$h(t) = -16t^2 + 64t + D \quad **$$

for some constant $D$. The height is initially 80 ft above the ground, so substituting $h(0) = 80$ shows

$$h(0) = -16(0)^2 + 64(0) + D = 80 \quad *$$
$$\implies D = 80\,.$$

Thus, the height is

$$h(t) = \boxed{-16t^2 + 64t + 80} \quad *$$

(c) The projectile reaches the ground when $h(t) = 0$:

$$-16t^2 + 64t + 80 = 0 \quad *$$
$$-16(t^2 - 4t - 5) = 0$$
$$-16(t-5)(t+1) = 0$$
$$\implies t = \boxed{5} \quad *$$

The solution $t = -1$ is extraneous because the domain in context is $\{t \mid t \geqslant 0\}$.

(d) The maximum height is attained at the moment the projectile is stationary in the air—that is, when $v(t) = 0$:

$$-32t + 64 = 0 \quad *$$

$$\implies t = 2\,. \quad *$$

[Because $h'(t) = v(t)$, this analysis is identical to locating the absolute maximum by determining the critical numbers.] Then the maximum height is

$$h(2) = -16(2)^2 + 64(2) + 80 = \boxed{144\,\text{ft}} \quad *$$

(e) The projectile strikes the ground at $t = 5$, so the velocity immediately before impact is

$$v(5) = -32(5) + 64 = -96\,\text{ft/sec}\,. \quad **$$

The *speed* is the absolute value of velocity, so the *speed* immediately before impact is

$$|v(5)| = \boxed{96\,\text{ft/sec}} \quad *$$

**13.** (a) With $n = 2$, the subintervals in the Midpoint Rule are $[1,3]$ and $[3,5]$. Then $\Delta x = \dfrac{5-1}{2} = 2$. Hence, the Midpoint Rule gives, for $f(x) = e^{-2x}$,

$$\int_1^5 e^{-2x}\,\mathrm{d}x \approx M_2$$

$$= [f(2) + f(4)]\,\Delta x \quad *$$

$$= \boxed{2\left(e^{-4} + e^{-8}\right)} \quad **$$

(b) With $n = 4$, the subintervals are $[1,2]$, $[2,3]$, $[3,4]$, and $[4,5]$. Thus, $\Delta x = \dfrac{5-1}{4} = 1$. With $f(x) =$

$e^{-2x}$, the left-endpoint approximation gives

$$\int_1^5 e^{-2x}\,\mathrm{d}x \approx L_4$$

$$= [f(1)+f(2)+f(3)+f(4)]\,\Delta x \quad *$$

$$= \boxed{\left(e^{-2}+e^{-4}+e^{-6}+e^{-8}\right)} \quad *$$

By the right-endpoint approximation,

$$\int_1^5 e^{-2x}\,\mathrm{d}x \approx R_4$$

$$= [f(2)+f(3)+f(4)+f(5)]\,\Delta x \quad *$$

$$= \boxed{\left(e^{-4}+e^{-6}+e^{-8}+e^{-10}\right)} \quad *$$

(c) If $n = 4$, then $\Delta x = \dfrac{5-1}{4} = 1$. Accordingly, Simpson's Rule gives, for $f(x) = e^{-2x}$,

$$\int_1^5 e^{-2x}\,\mathrm{d}x \approx S_4$$

$$= \frac{\Delta x}{3}[f(1)+4f(2)+2f(3)+4f(4)+f(5)] \quad *$$

$$= \boxed{\frac{1}{3}\left(e^{-2}+4e^{-4}+2e^{-6}+4e^{-8}+e^{-10}\right)} \quad **$$

(d) The derivatives of $f(x) = e^{-2x}$ are

$$f'(x) = -2e^{-2x} \qquad f''(x) = 4e^{-2x} \qquad f'''(x) = -8e^{-2x} \qquad f^{(4)}(x) = 16e^{-2x}. \quad *$$

The maximum value of $\left|f^{(4)}(x)\right| = 16e^{-2x}$ on $[1,5]$ is

$$M_4 = 16e^{-2}. \quad *$$

Thus, an error bound is

$$|E_S| = \frac{M_4(b-a)^5}{180n^4}$$

$$= \frac{16e^{-2}(5-1)^5}{180(4)^4} \quad *$$

$$= \boxed{\frac{16}{45e^2}} \quad **$$

**14.** Remember that definite integrals are simply numbers, so changing the variable of integration does not alter the integral's value; for example, $\int_1^2 f(t)\,\mathrm{d}t = \int_1^2 f(x)\,\mathrm{d}x$.

(a) Computing the area under the curve $f'$ from $t=-2$ to $t=2$, we see

$$g(2) = \int_{-2}^{2} f'(t)\,\mathrm{d}t = \tfrac{1}{2}\pi(2)^2 = \boxed{2\pi} \quad *$$

In addition,

$$g(-2) = \int_{-2}^{-2} f'(t)\,\mathrm{d}t = \boxed{0} \quad *$$

Lastly, referencing the area of the triangle between $t=-6$ and $t=-2$ shows

$$g(-6) = \int_{-2}^{-6} f'(t)\,\mathrm{d}t = -\int_{-6}^{-2} f'(t)\,\mathrm{d}t = \tfrac{1}{2}(4)(2) = \boxed{4} \quad *$$

(b) By the Net Change Theorem,

$$f(4) = f(-2) + \int_{-2}^{4} f'(t)\,\mathrm{d}t \quad *$$

$$= 5 + \int_{-2}^{4} f'(t)\,\mathrm{d}t$$

$$= 5 + (2\pi + 2)$$

$$= \boxed{7 + 2\pi} \quad *$$

(The area under the curve $y = f'(t)$ is a semicircle of area $2\pi$ on $[-2,2]$ and a triangle of area 2 on $[2,4]$.)

(c) By Part I of the Fundamental Theorem of Calculus,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-2}^{x} f'(t)\,\mathrm{d}t = f'(x).$$ *

Then $g''(x) = f''(x)$. The graph of $f$ has an inflection point when $f''$ changes signs—that is, when the graph of $f'$ has a turning point. At 0, the graph of $f'$ changes from increasing to decreasing; at 2, the graph changes from decreasing to increasing. The inflection points are therefore

$$x = \boxed{0} \qquad \text{and} \qquad x = \boxed{2}$$ **

(d) Observe that $f'(3) = 1$ and $f'(6) = 4$. By Part II of the Fundamental Theorem of Calculus,

$$\int_3^6 \left[2f''(x) - 4\right]\,\mathrm{d}x = \left[2f'(x) - 4x\right]\Big|_3^6$$ *

$$= 2[f'(6) - f'(3)] - 4(6-3)$$

$$= 2(4-1) - 4(6-3)$$ *

$$= \boxed{-6}$$ *

(e) On $-6 < t < -2$, the graph of $f'$ is linear with a slope of $\frac{1}{2}$ and $t$-intercept at $t = -2$. An equation for $f'$ on this interval is (by the point-slope form of a line)

$$f'(t) = \tfrac{1}{2}(t+2) = \tfrac{1}{2}t + 1\,.$$ *

Antidifferentiating yields

$$f(t) = \tfrac{1}{4}t^2 + t + C$$ *

for some constant $C$. Substituting the given condition $f(-2) = 5$ gives

$$f(-2) = \tfrac{1}{4}(-2)^2 + (-2) + C = 5$$ *

$$\implies C = 6\,.$$

Accordingly,

$$f(t) = \tfrac{1}{4}t^2 + t + 6\,.$$

The variable has no effect on the function's identity, so we are free to write

$$f(x) = \boxed{\frac{1}{4}x^2 + x + 6}$$

*