VALSCALC

Chapter 0 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 0: Preliminaries.

0.1: Fundamental Skills in Algebra
0.2: Numbers, Sets, and Absolute Values
0.3: Coordinates and Geometry
0.4: Defining a Function
0.5: Linear Functions and Equations
0.6: Modifying Functions; Inverse Functions
0.7: Quadratics
0.8: Trigonometry
0.9: Exponents and Logarithms
0.10: Sigma Notation

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Preliminaries

Number of Questions—19

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points Earned	Points Available
Short Questions I		27
Short Questions II		28
Question 17		15
Question 18		15
Question 19		15
TOTAL		**100**

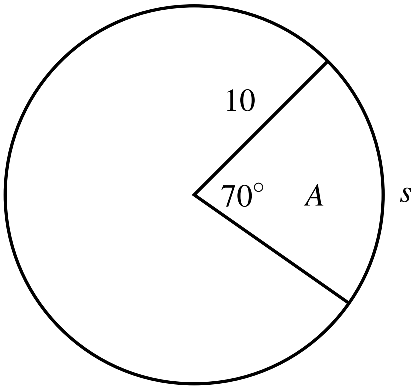
Short Questions I

1. Determine the domain of $v(x) = \sqrt{\frac{1}{4}(x-8)+1}$. *(3 pts.)*

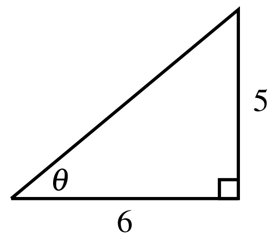
2. Factor $27x^5 + 8x^2$ completely. *(3 pts.)*

3. Determine the solution interval to the inequality $5|x-4| < 20$. *(3 pts.)*

4. In the following circular sector, calculate the arc length s and the area A. *(3 pts.)*



5. In the following triangle, obtain expressions for $\csc\theta$, $\cot\theta$, and θ. *(3 pts.)*



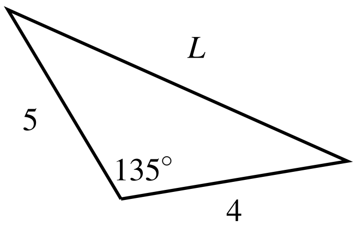
6. A horizontal ellipse is centered at $(-5, 2)$ with width 10 and height 8. Write an equation for the ellipse in the xy-plane. *(3 pts.)*

7. Determine the solution to the following system of equations: *(3 pts.)*

$$\begin{cases} 2x + y = 8 \\ x - 2y = 4\,. \end{cases}$$

8. For what value of c are the lines $4x + y = 2$ and $y = cx$ parallel? For what value are they perpendicular? Justify your reasoning. *(3 pts.)*

9. In the following triangle, determine the exact value of L. Evaluate any trigonometric expressions. (*3 pts.*)



Short Questions II

10. Rewrite $y = |4x - 8|$ as a piecewise function. *(4 pts.)*

11. Evaluate $\sum_{i=1}^{8}(2i - 5)$. *(4 pts.)*

12. Rewrite the following expression as a single logarithm: *(4 pts.)*

$$\log_4 x + 2\log_4 y - \tfrac{1}{3}\log_4 z,$$

where x, y, and z are positive.

13. Assuming all quantities are positive, simplify the expression *(4 pts.)*

$$\frac{\left(16x^4y^8z^2\right)^{1/2}}{\left(2xy^3z^{-4}\right)^2}.$$

14. Determine the solutions to the equation $e^{2x} - 7e^x - 8 = 0$. Express your answers using the natural logarithm. *(4 pts.)*

15. If $\log_3 x + \log_3(x+2) = 1$, then calculate x. *(4 pts.)*

16. To what integer does $\dfrac{\sin 4x}{(\cos^2 x - \sin^2 x)\sin 2x}$ simplify, assuming no division by 0 ? *(4 pts.)*

Long Questions

17. Consider the two rational functions $f(x) = \dfrac{x}{x+2}$ and $g(x) = \dfrac{4}{x^2-4}$.

(a) Find the domains of f and g. Write your answers in interval notation. *(3 pts.)*

(b) At what value of x are $f(x)$ and $g(x)$ both undefined? *(2 pts.)*

(c) Evaluate $\dfrac{f(x)}{g(x)}$ and $(f+g)(x)$. Write each answer as a single rational expression. *(4 pts.)*

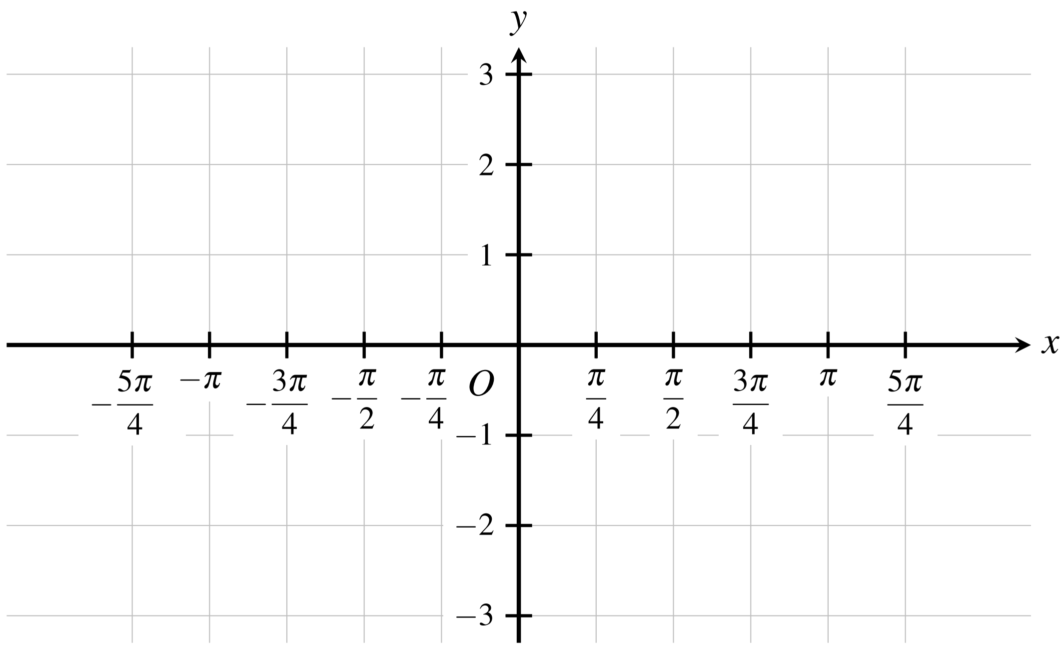
(d) Evaluate $g(f(-3))$ and $g(g(0))$. *(4 pts.)*

(e) Is the function g even, odd, or neither? Justify your answer. *(2 pts.)*

18. Consider the function $f(x) = -2\sin 4x + 1$.

(a) Using interval notation, write the domain and range of f. (2 pts.)

(b) Sketch the graph of $y = f(x)$ in the following figure. (5 pts.)



(c) Determine $f^{-1}(x)$, the inverse function of $f(x)$. Assume that the domain of f is restricted to $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$, an interval on which f is one-to-one. *(4 pts.)*

(d) Write the domain and range of $f^{-1}(x)$ using interval notation. *(3 pts.)*

(e) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other across a line ℓ that passes through the origin. What is the slope of ℓ ? *(1 pt.)*

19. Consider the family of functions $q(x) = 4x^2 + kx + 3$ for some constant k.

(a) For $k = -7$, find the x-intercepts and y-intercept of the graph of $y = q(x)$. *(4 pts.)*

(b) For $k = 8$, rewrite $q(x)$ in vertex form. What is the vertex, and does it correspond to the lowest or highest point on the graph of $y = q(x)$? *(4 pts.)*

(c) A new function is defined such that $r(x) = 4 - x$. For $k = 1$, determine the values of x such that $q(x) = r(x)$. *(3 pts.)*

(d) Determine the values of k for which $q(x)$ has no real zeros. *(4 pts.)*

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions I (3 points each)

1. The argument of the square root must be nonnegative:

$$\tfrac{1}{4}(x-8)+1 \geqslant 0 \quad *$$

$$\frac{x}{4} \geqslant 1$$

$$x \geqslant 4\,. \quad *$$

Thus, the domain is

$$\boxed{[4,\infty)} \quad *$$

2. First we extract the greatest common factor (GCF), which is x^2, to get

$$x^2(27x^3+8)\,. \quad *$$

The expression in parentheses is a sum of cubes: $(3x)^3+2^3$. Thus,

$$x^2(27x^3+8) = x^2(3x+2)(9x^2-6x+4)\,. \quad **$$

3. Dividing both sides by 5 gives

$$|x-4| < 4\,.$$

The inequality is satisfied by $(x-4)<4$ and $-(x-4)<4$. In the first case,

$$x-4<4$$

$$x<8\,. \quad *$$

In the second case,

$$-(x-4)<4$$

$$x-4>-4$$

$$x>0\,. \quad *$$

Thus, the solution interval is

$$\{x \mid x > 0 \text{ and } x < 8\} = \boxed{(0,8)}$$ *

4. The measure of the central angle in radians is

$$70^\circ \times \frac{\pi}{180^\circ} = \frac{7\pi}{18}.$$ *

The arc length is

$$s = 10\left(\frac{7\pi}{18}\right) = \boxed{\frac{35\pi}{9}}$$ *

Additionally, the area is

$$A = \frac{1}{2}\left(\frac{7\pi}{18}\right)(10)^2 = \boxed{\frac{175\pi}{9}}$$ *

5. By the Pythagorean Theorem, the hypotenuse length is

$$\sqrt{6^2+5^2} = \sqrt{61}.$$

The cosecant is the ratio of the hypotenuse's length to the opposite side's length, so

$$\csc\theta = \boxed{\frac{\sqrt{61}}{5}}$$ *

The cotangent is the ratio of the adjacent side length to the opposite side length:

$$\cot\theta = \boxed{\frac{6}{5}}$$ *

The angle θ is given by

$$\theta = \boxed{\tan^{-1}\left(\frac{5}{6}\right)} = \boxed{\sin^{-1}\left(\frac{5}{\sqrt{61}}\right)}$$ *

(There are several correct answers using inverse trigonometric functions.)

6. The horizontal ellipse is modeled by the equation

$$\frac{(x+5)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1\,.$$ *

The width is $2a = 10$, so $a = 5$. Additionally, the height is $2b = 8$, from which $b = 4$. Thus, an equation of the ellipse is

$$\boxed{\frac{(x+5)^2}{25} + \frac{(y-2)^2}{16} = 1}$$ **

7. It is easiest to solve this system of equations by elimination. For example, one method is to multiply the second equation $(x - 2y = 4)$ by 2. The system then becomes

$$\begin{cases} 2x + y = 8 \\ 2x - 4y = 8\,. \end{cases}$$

Subtracting the second equation from the first equation yields

$$5y = 0 \implies y = 0\,.$$ *

Then by substituting back, we find

$$2x + 0 = 8 \implies x = 4\,.$$ *

The solution is therefore

$$\boxed{(4,0)}$$ *

8. Solving for y in $4x + y = 2$ gives

$$y = -4x + 2\,.$$ *

Thus, this line has slope -4. Conversely, the line $y = cx$ has slope c. Two lines are parallel if their slopes are equal, and two lines are perpendicular if their slopes are negative reciprocals of each other.

Accordingly, the answers are as follows:

$$\text{parallel: } c = \boxed{-4} \qquad *$$

$$\text{perpendicular: } c = \boxed{\frac{1}{4}} \qquad *$$

9. By the Law of Cosines,

$$L^2 = (4)^2 + (5)^2 - 2(4)(5)\cos 135^\circ\,. \qquad *$$

Since $\cos 135^\circ = -\dfrac{\sqrt{2}}{2}$, the preceding equation becomes

$$L^2 = 41 + 20\sqrt{2}\,. \qquad *$$

Taking the positive square root gives

$$L = \boxed{\sqrt{41 + 20\sqrt{2}}} \qquad *$$

Short Questions II (4 points each)

10. The expression $|4x-8|$ is $(4x-8)$ if $x \geqslant 2$ and $-(4x-8)$ if $x < 2$. We note the following:

$$|4x-8| = 4x-8 \quad \text{if } x \geqslant 2\,, \qquad *$$

$$|4x-8| = -(4x-8) \text{ if } x < 2\,. \qquad *$$

Thus,

$$|4x-8| = \boxed{\begin{cases} 4x-8 & x \geqslant 2 \\ -4x+8 & x < 2 \end{cases}} \quad \text{or} \quad |4x-8| = \boxed{\begin{cases} 4x-8 & x > 2 \\ -4x+8 & x \leqslant 2 \end{cases}} \qquad **$$

11. We first split the sum and then use summation properties, as follows:

$$\sum_{i=1}^{8}(2i-5) = 2\sum_{i=1}^{8} i - \sum_{i=1}^{8} 5 \qquad *$$

$$= 2\left(\frac{8(8+1)}{2}\right) - 8(5) \qquad **$$

$$= \boxed{32} \qquad *$$

12. We have

$$\log_4 x + 2\log_4 y - \tfrac{1}{3}\log_4 z = \log_4 x + \log_4\left(y^2\right) - \log_4\left(z^{1/3}\right) \qquad **$$

$$= \log_4\left(\frac{xy^2}{z^{1/3}}\right) \qquad **$$

$$= \boxed{\log_4\left(\frac{xy^2}{\sqrt[3]{z}}\right)}$$

13. The expression is

$$\begin{aligned}\frac{\left(16x^4y^8z^2\right)^{1/2}}{(2xy^3z^{-4})^2} &= \frac{\sqrt{16x^4y^8z^2}}{(2xy^3z^{-4})^2} \\ &= \frac{4x^2y^4z}{4x^2y^6z^{-8}} && ** \\ &= \frac{1}{y^2z^{-9}} && ** \\ &= \boxed{\frac{z^9}{y^2}}\end{aligned}$$

14. The equation $e^{2x} - 7e^x - 8 = 0$ is quadratic in e^x since $e^{2x} = (e^x)^2$. Factoring yields

$$(e^x - 8)(e^x + 1) = 0. \qquad **$$

Equating each factor to 0, we see

$$e^x - 8 = 0 \implies e^x = 8 \implies x = \ln 8. \qquad *$$

The equation $e^x + 1 = 0$ has no solutions because the range of e^x is $(0, \infty)$. Hence, the only solution is

$$x = \boxed{\ln 8} \qquad *$$

15. Both logarithms share the same base, so combining the sum produces

$$\log_3\left(x^2 + 2x\right) = 1. \qquad *$$

Exponentiating both sides using the base 3 gives

$$3^{\log_3\left(x^2+2x\right)} = 3^1$$

$$x^2+2x = 3 \quad *$$

$$x^2+2x-3 = 0$$

$$(x+3)(x-1) = 0 \quad *$$

$$\implies x = -3\,, x = 1\,.$$

But the domain of $\log_3 x + \log_3(x+2)$ is $(0,\infty)$ (to avoid nonpositive arguments), so $x=-3$ is extraneous. Instead, the only solution is

$$x = \boxed{1} \quad *$$

16. Because $\sin 4x = 2\sin 2x\cos 2x$ and $\cos^2 x - \sin^2 x = \cos 2x$, we have

$$\frac{\sin 4x}{\left(\cos^2 x - \sin^2 x\right)\sin 2x} = \frac{2\sin 2x\cos 2x}{\left(\cos^2 x - \sin^2 x\right)\sin 2x} \quad *$$

$$= \frac{2\sin 2x\cos 2x}{\cos 2x\sin 2x} \quad *$$

$$= \boxed{2} \quad **$$

Long Questions (15 points each)

17. (a) In each function, we exclude x-values that result in a denominator of 0. For example, the denominator of $f(x)$ is 0 when $x = -2$, so the domain is all x *excluding* -2—namely,

$$\{x \mid x \neq -2\} = \boxed{(-\infty, -2) \cup (-2, \infty)}$$ *

The denominator of $g(x)$ is 0 when

$$\begin{aligned} x^2 - 4 &= 0 \\ (x+2)(x-2) &= 0 \\ \implies x &= -2, 2. \end{aligned}$$ *

Hence, the domain of $g(x)$ is

$$\{x \mid x \neq -2 \text{ and } x \neq 2\} = \boxed{(-\infty, -2) \cup (-2, 2) \cup (2, \infty)}$$ *

(b) Both domains exclude the point

$$x = \boxed{-2}$$ **

Thus, both functions are undefined at $x = -2$.

(c) The quotient is

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{x/(x+2)}{4/(x^2-4)} \\ &= \frac{x}{x+2} \cdot \frac{x^2-4}{4} \\ &= \frac{x}{x+2} \cdot \frac{(x+2)(x-2)}{4} \quad * \\ &= \frac{x\cancel{(x+2)}(x-2)}{4\cancel{(x+2)}} \\ &= \boxed{\frac{x(x-2)}{4}} \quad * \end{aligned}$$

where $x \neq -2, 2$. Conversely, to compute the sum we attain a common denominator. The easiest way to do so is to multiply the numerator and denominator of $f(x)$ each by $(x-2)$. Doing so shows

$$\begin{aligned}(f+g)(x) &= \frac{x}{x+2}+\frac{4}{x^2-4}\\ &= \frac{x}{x+2}\cdot\frac{x-2}{x-2}+\frac{4}{x^2-4} && *\\ &= \frac{x(x-2)}{x^2-4}+\frac{4}{x^2-4}\\ &= \boxed{\frac{x(x-2)+4}{x^2-4}} && *\end{aligned}$$

(d) Note that

$$f(-3)=\frac{(-3)}{(-3)+2}=3\,. \qquad *$$

Thus,

$$g(f(-3))=g(3)=\frac{4}{(3)^2-4}=\boxed{\frac{4}{5}} \qquad *$$

Likewise,

$$g(0)=\frac{4}{(0)^2-4}=-1\,. \qquad *$$

Accordingly,

$$g(g(0))=g(-1)=\frac{4}{(-1)^2-4}=\boxed{-\frac{4}{3}} \qquad *$$

(e) Observe that

$$g(-x)=\frac{4}{(-x)^2-4}=\underbrace{\frac{4}{x^2-4}}_{g(x)}\,.$$

Because $g(-x)=g(x)$,

$$g \text{ is } \boxed{\text{even}} \qquad **$$

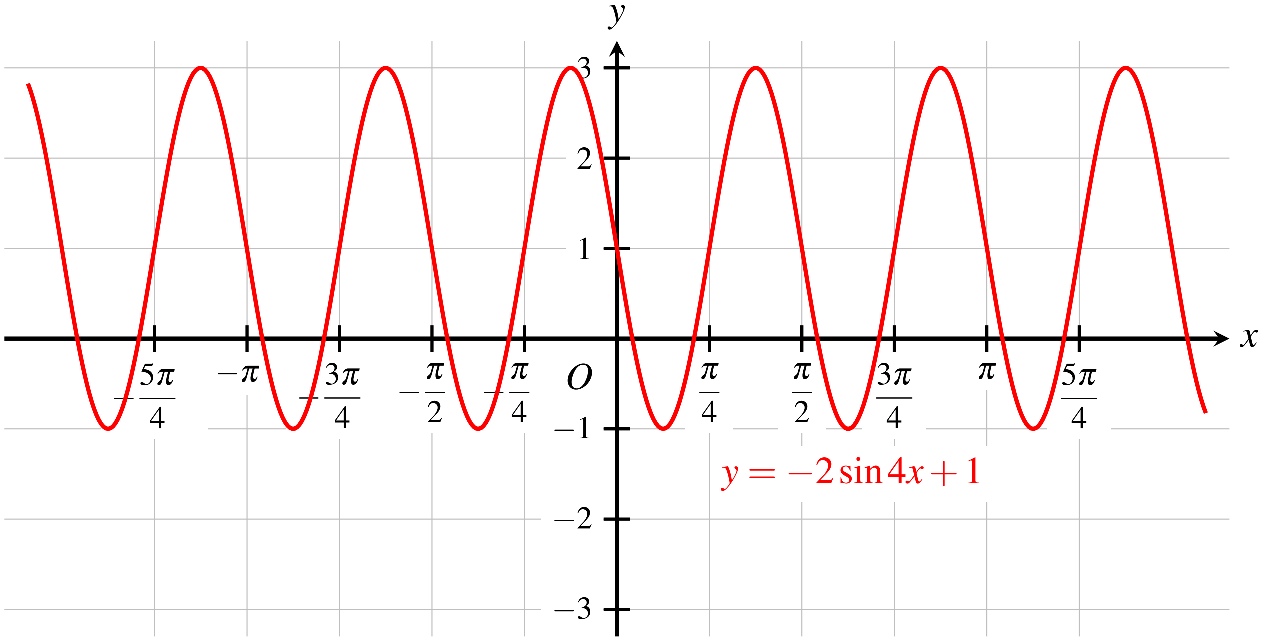
18. (a) The domain of any sinusoidal function is $(-\infty, \infty)$. The range of sine is $[-1, 1]$; but the graph of $f(x) = -2\sin 4x + 1$ is dilated vertically by a factor of 2 and shifted up 1, so the range is $[-1, 3]$. So the answers are as follows:

$$\text{domain: } (-\infty, \infty)\,, \quad *$$

$$\text{range: } [-1, 3]\,. \quad *$$

(b) The graph has the following features:

- A y-intercept of $(0, 1)$ *
- A midline of $y = 1$ *
- An amplitude of 2 *
- A period of $\dfrac{\pi}{2}$ *
- Decreasing at $x = 0$ *



(c) If f is restricted to the domain $\left[-\dfrac{\pi}{8}, \dfrac{\pi}{8}\right]$, then we solve for the inverse function by replacing $f(x)$

with y, swapping x and y, and solving for y, as follows:

$$x = -2\sin 4y + 1$$ *

$$\sin 4y = \tfrac{1}{2}(1-x)$$

$$4y = \sin^{-1}\left[\tfrac{1}{2}(1-x)\right]$$ *

$$y = \tfrac{1}{4}\sin^{-1}\left[\tfrac{1}{2}(1-x)\right].$$ *

Thus, the inverse function is

$$f^{-1}(x) = \boxed{\tfrac{1}{4}\sin^{-1}\left[\tfrac{1}{2}(1-x)\right]}$$ *

(d) The range of $f(x) = -2\sin 4x + 1$ is $[-1,3]$, so for $f^{-1}(x)$,

domain: $\boxed{[-1,3]}$ **

In addition, the inverse function exists when the domain of f is $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$, so for $f^{-1}(x)$,

range: $\boxed{\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]}$ *

(e) A function f and its inverse f^{-1} are reflections of each other across the line $y = x$. Thus, the slope of ℓ is

$$\boxed{1}$$ *

19. (a) For $q(x) = 4x^2 - 7x + 3$, the y-intercept is found by substituting $x = 0$:

$$q(0) = 4(0)^2 - 7(0) + 3 = 3\,.$$

Thus, the y-intercept is

$$\boxed{(0,3)}$$ *

To find the x-intercepts, we solve $q(x) = 0$—that is, $4x^2 - 7x + 3 = 0$. To factor the left side, we note that two numbers whose sum is -7 and whose product is $4 \times 3 = 12$ are -4 and -3. Thus, we rewrite $-7x$ as $-4x - 3x$ and factor by grouping to attain

$$4x^2 - 4x - 3x + 3 = 0$$ *

$$4x(x-1) - 3(x-1) = 0$$ *

$$(4x-3)(x-1) = 0$$

$$\implies x = \frac{3}{4}, 1\,.$$

Thus, the x-intercepts are

$$\boxed{\left(\frac{3}{4}, 0\right)} \quad \text{and} \quad \boxed{(1,0)}$$ *

(b) In $q(x) = 4x^2 + 8x + 3$, factoring the leading coefficient (4) gives

$$q(x) = 4\left(x^2 + 2x + \frac{3}{4}\right).$$

To complete the square, we add and subtract $(2/1)^2 = 1$ in the parentheses, as follows:

$$q(x) = 4\left(x^2 + 2x + 1 - 1 + \frac{3}{4}\right)$$ *

$$= 4\left[(x+1)^2 - \frac{1}{4}\right]$$

$$= \boxed{4(x+1)^2 - 1}$$ *

The vertex is

$$\boxed{(-1,-1)}$$ *

Because the leading coefficient of q is positive, the parabola opens upward. Thus,

$$\text{the vertex is the } \boxed{\text{lowest point}} \text{ on the graph of } y = q(x)\,.$$ *

(c) We seek the solutions to

$$4x^2 + x + 3 = 4 - x\,.$$

Moving all terms to the left gives

$$4x^2 + 2x - 1 = 0\,.$$

Factoring fails, so we use the Quadratic Formula. With $a = 4$, $b = 2$, and $c = -1$, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2(4)} \quad * \\ &= \boxed{\frac{-2 \pm \sqrt{20}}{8}} \quad ** \end{aligned}$$

Because $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$, the answer may be simplified to

$$x = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}\,.$$

(d) With $a = 4$, $b = k$, and $c = 3$, the discriminant is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= k^2 - 4(4)(3) \quad * \\ &= k^2 - 48\,. \end{aligned}$$

The function $q(x)$ has no real zeros if $\Delta < 0$. We see

$$k^2 - 48 = 0 \implies k = \pm\sqrt{48}\,.$$ *

For $-\sqrt{48} < k < \sqrt{48}$, we see $k^2 - 48 < 0$. Thus, q has no real zeros for

$$\boxed{-\sqrt{48} < k < \sqrt{48}}$$ **