Integration

Integrals of Key Functions

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C,\quad b \neq 1$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

Integrals of Trigonometric Functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

Rules of Integrals

Let c be a constant.

$$\int [cf(x)]\,dx = c\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Rules of Definite Integrals

Let c be a constant.

$$\int_a^b c\,dx = c(b-a) \qquad \int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Riemann Definition of a Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

For equal subinterval widths,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,,$$

where $x_i = a + i\Delta x$ and $\Delta x = \dfrac{b-a}{n}$.

Comparison Properties of Definite Integrals

- If $f(x) \geqslant 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geqslant 0$.
- If $f(x) \geqslant g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.
- If $m \leqslant f(x) \leqslant M$ on $[a,b]$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

Fundamental Theorem of Calculus, Part I

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x))u'(x) - f(v(x))v'(x)$$

Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$ and $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Net Change Theorem

If f' is continuous on $[a,b]$, then

$$f(b) = f(a) + \int_a^b f'(x)\,dx$$

Substitution Rule (u-Substitution)

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integration

Integrals of Odd and Even Functions

- If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
- If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Integration by Parts

For differentiable functions u and v,

$$\int u\,dv = uv - \int v\,du$$

Integrating $\int \sin^m x \cos^n x\,dx$

- For odd m, substitute $u = \cos x$ in
$$\int \sin^m x \cos^n x\,dx = \int \left(1-\cos^2 x\right)^{(m-1)/2} \cos^n x \sin x\,dx$$
- For even m and odd n, substitute $u = \sin x$ in
$$\int \sin^m x \cos^n x\,dx = \int \sin^m x \left(1-\sin^2 x\right)^{(n-1)/2} \cos x\,dx$$
- For even m and even n, use the power-reduction formulas
$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

Integrating $\int \sec^m x \tan^n x\,dx$

- For even m, substitute $u = \tan x$ in
$$\int \sec^m x \tan^n x\,dx = \int \left(\tan^2 x + 1\right)^{(m-2)/2} \tan^n x \sec^2 x\,dx$$
- For odd n, substitute $u = \sec x$ in
$$\int \sec^m x \tan^n x\,dx = \int \sec^{m-1} x \left(\sec^2 x - 1\right)^{(n-1)/2} \sec x \tan x\,dx$$

Integrating $\int \csc^m x \cot^n x\,dx$

- For even m, substitute $u = \cot x$ in
$$\int \csc^m x \cot^n x\,dx = \int \left(\cot^2 x + 1\right)^{(m-2)/2} \cot^n x \csc^2 x\,dx$$
- For odd n, substitute $u = \csc x$ in
$$\int \csc^m x \cot^n x\,dx = \int \csc^{m-1} x \left(\csc^2 x - 1\right)^{(n-1)/2} \csc x \cot x\,dx$$

Integrating Products of Sines and Cosines

Integrals of products of sines and cosines can be reexpressed through product-to-sum identities:

$$\sin A \sin B = \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A \cos B = \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Trigonometric Substitutions

Radicals can be stripped using the following substitutions.

Radical	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$	$\tan^2\theta + 1 = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Integration by Partial Fractions

Rational functions whose denominator has a higher degree than the numerator can be decomposed as follows.

Term in Denominator	Partial Fraction Form
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^n$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^n$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

Type I Improper Integrals

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

If both integrals converge, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

p-Integrals

$\int_1^{\infty} \dfrac{1}{x^p}\,dx$ diverges for $p \leqslant 1$ and converges for $p > 1$.

Integration

Type II Improper Integrals

- If f is continuous on $[a, b)$ and has a vertical asymptote at $x = b$, then
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
- If f is continuous on $(a, b]$ and has a vertical asymptote at $x = a$, then
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$
- If f has a discontinuity at c $(a < c < b)$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
if both integrals on the right converge; otherwise, $\int_a^b f(x)\,dx$ diverges.

Comparison Properties for Improper Integrals

- If $0 \leqslant g(x) \leqslant f(x)$ on $[a, \infty)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.
- If $0 \leqslant f(x) \leqslant g(x)$ on $[a, \infty)$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

Endpoint Approximations

- Left-endpoint approximation:
$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^n f(x_{i-1})\,\Delta x$$
- Right-endpoint approximation:
$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^n f(x_i)\,\Delta x$$

$$\Delta x = \frac{b-a}{n} \qquad |E_n| \leqslant \frac{M_1(b-a)^2}{2n}$$

where $|f'(x)| \leqslant M_1$ on $[a, b]$.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^n f(\overline{x}_i)\,\Delta x$$

$$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) \qquad \Delta x = \frac{b-a}{n}$$

$$|E_M| \leqslant \frac{M_2(b-a)^3}{24n^2}$$

where $|f''(x)| \leqslant M_2$ on $[a, b]$.

Trapezoidal Rule

$$\begin{aligned}\int_a^b f(x)\,dx &\approx T_n \\ &= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]\end{aligned}$$

$$\Delta x = \frac{b-a}{n}$$

$$|E_T| \leqslant \frac{M_2(b-a)^3}{12n^2}$$

where $|f''(x)| \leqslant M_2$ on $[a, b]$.

Simpson's Rule

For even n,

$$\begin{aligned}\int_a^b f(x)\,dx &\approx S_n \\ &= \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \\ &\qquad + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]\end{aligned}$$

$$\Delta x = \frac{b-a}{n}$$

$$|E_S| \leqslant \frac{M_4(b-a)^5}{180n^4}$$

where $\left|f^{(4)}(x)\right| \leqslant M_4$ on $[a, b]$.